ACCELERATED LIFE TEST PLANS

DISSERTATION

submitted in partial fulfilment of the requirements
for the award of the degree of

Master of Philosophy

IN

STATISTICS

BY

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Under the Supervision of

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DEDICATED

TO MY

BELOVED PARENTS

IN PARTIAL FULFILMENT

TO WHAT I OWE THEM

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CERTIFICATE

Certified that **MR. AIJAZ AHMAD HAKAK** has carried out research on **"ACCELERATED LIFE TEST PLANS"** under my supervision and the work is suitable for submission for the award of the degree of **MASTER OF PHILOSOPHY** in **STATISTICS.**

2/5/95

(DR. ARIF-UL-ISLAM)
Supervisor

PREFACE

Starting in the early 1950s, the word reliability acquired a highly specialized technical meaning in relation to the control of quality of manufactured product. As per the official definition of the Electronics Industries Association (EIA), quoted in "Reliability Principles & Practices" by S.R. Calabro. The reliability is, "the Probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered." The interest in Reliability theory currently exhibited by Engineers, mathematicians, economists, industrial managers and those concerned with the environmental and life sciences has stimulated the research work in this field. Electrical, Electronic and Mechanical equipment is being increasingly used in a number of fields - in industry for control of processes, in computers, in Medical Electronics, Atomic Energy, Communications, navgation at sea and in the air and many other fields. It is essential that this equipment should operate reliably under all the conditions in which it is used. However, the more reliable a device is, the more difficult it is to measure its reliability. This is so because many years of testing under actual operating conditions would be required to obtain numerical measures of its reliability. Even if such testing were feasible, the rate of technical advance is so great that parts would be obsolete by the time their reliability had been measured. In additions, many of the components used in practice are subjected environments that are difficult to stimulate in the laboratory. One approach to solving this predicament is to use accelerated life tests.

Accelerated testing consists of a variety of test methods for shortening the life of products or hastening the degradation of their performance. The aim of such testing is to quickly obtain data which, properly modeled and analyzed, yield desired information on product life or performance under normal use. Such testing saves much time and money.

This manuscript is intended to present a survey, of available literature on "ACCELERATED LIFE TEST PLANS". The work has been divided into four chapters with a comprehensive list of references at the end. The references are arranged authorwise.

The **Chapter-I** entitled **"BASIC CONCEPT OF RELIABILITY THEORY"** is of an introductory nature. As the title signifies, the chapter deals with some concepts of reliability which are to be used subsequently.

The **Chapter-II** entitled **"DEVELOPMENT OF ALT SAMPLING PLANS FOR EXPONENTIAL DISTRIBUTION"** presents a procedure for developing life test sampling plans for exponential distributions based upon accelerated life testing (ALT). Type-II censoring is assumed at each overstress level. Simplified formulae are given for computing type I and type-II error probabilities in a two stress level accelerated life test plan developed by Yum and Kim (1990). Using the new formulae, some acceptance sampling plans for the mean value of an exponential distribution, based on type-II censored data are obtained.

The **Chapter-III** entitled **"A SAMPLING PLAN FOR SELECTING THE MOST RELIABLE PRODUCT UNDER THE ARRHENIUS ACCELERATED LIFE TEST MODELS"** deals with the selection of most reliable product. The decision maker often faces a problem of selecting the most reliable product from among several competing products mainly because it may take a long time to observe failures under normal operating condition. To shorten the life-testing time, an ALT is usually used. For life-stress relations following a Weibull-Arrhenius model, this chapter proposes an MLR (Modified Likelihood Ratio) rule to select the most reliable product.

The **Chapter-IV** entitled **"PLANNING ACCELERATED LIFE TESTS FOR SELECTING THE MOST RELIABLE PRODUCT"** proposes a systematic approach to the selection problem for highly reliable products which possess a Weibull distribution whose characteristic life is a log-linear function of stress. First optimum test plans for both type-I and type-II censoring are derived by minimizing the asymptotic variance of estimated quantiles at the design stress. Next, an intutively appealing selection rule is proposed. The sample size and censoring time needed by this selection rule are computed with a predetermined time-saving factor and a minimum probability of correct selection (CS). Finally, a cost criterion is used to compare these two censoring plans.

ACKNOWLEDGEMENT

This dissertation entitled, **"ACCELERATED LIFE TEST PLANS,"** is submitted to the Department of Statistics and Operations Research, Aligarh Muslim University, (Aligarh) in partial fulfilment for the award of the degree of **MASTER OF PHILOSOPHY** in Statistics.

Praise be to Allah, Lord of the Universe, the Most Gracious. Most Mercifull who bestowed me with utmost courage and wisdom to complete this work.

I would like to express my indebtness and sincere gratitude to my worthy supervisor, **Dr. Arif-ul-Islam**, for his excellent supervision, invaluable advice, encouragement, timely help and necessary assistance provided through all the stages of writing this dissertation. His personal care made me emboldened to complete this work. I feel proud in expressing my profound gratitude to him. It is my ardent belief that his efforts would not culminate here but would further be reflected in the endeavours to come.

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I am very thankful to all the teaching staff of my department whose valuable suggestion boasted up my morale. The generosity, wisdom, knowledge and suggestions of **Prfo. Sirajur Rahman** and **Prof. Abdul Hameed Khan**, have been of invaluable help to me.

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The scholar had to visit many institutes located in Delhi to collect the research material. I shall be failing in my duty if I won't mention Swadesh Kumar Singh (PG Student of IIT) and Vijay Nagendra Yadav (PG Student of ISI) for their hospitality during my stay in their respective institutes.

I owe a special debt of gratitude to my friends - cum - roommates namely, Naushad Adil Azad - Research Scholar, Department of History, Mohd. Obaid Alam - B.Sc. (Engg.) Student, Hasnain Sarshar - Student of M.B.B.S. and Akmal Yazdani for their unequivocable help. They used to encourage me every time and provided congenial atmosphere for my study. Such a gesture of affection and goodwill is unforgetable. May Allah grant them success in their relevant fields.

As is said there are foes as well as friends. The foes created manifold problems, but the help that I received from friends gave me the confidence to take challenges head on. To take a noble revenge, I desist from disclosing the names of such foes. Instead I thank them for their jealousy because their very envious nature inspired me a lot.

Last but not the least I would like to thank my parents whose encouragement, moral support, prayers and financial assistance enabled me to persue studies. I thank them for enduring the months of isolation generated by the preparation of this work. I would like to record my thanks to my brothers namely Ishtiaq Ahmad Hakak and Irshad Ahmad Hakak for their affectionate love.

(AIJAZ AHMAD HAKAK)

CONTENTS

CHAPTER - I

BASIC CONCEPTS OF RELIABILITY THEORY

1.1 INTRODUCTION

In the light of rapid technological advances, the modern products have become increasingly more sophisticated. Very detailed and complex equipment has been researched, designed, developed and implemented for space exploration, military applications, and commercial uses. In general, each piece of equipment is composed of numerous elementary components and/or sub-systems that work as a unit either to achieve specific objectives or to perform a variety of functions. As a consequence increasing attention has been focused on the evaluation of whether a given device successfully performs its intended function. Reliability analyses evaluate the performance of equipment.

In particular, we use the term reliability to mean the probability that a piece of equipment (component, subsystem, or system) successfully performs its intended function for a given period of time under specified conditions. Now this involves several ingredients. The product is designed and manufactured to be used over some field of application. For example, a chain saw is for cutting wood, green or dry, hard or soft, but not intended for cutting metal which may occur in the wood. Nevertheless, some safeguards can and are built into a chain saw to protect the user against moderate misuse should it occur. Some designs are used to make product not only fool proof but "damn-fool proof".

Another facet of reliability is the time factor or repetitive usage. A product must be designed and produced so that it will perform its function for at least a minimum length of life, or a minimum number of cycles, such as startings. These guaranteed lengths of life may be short or long, but are an integral part of the picture.

Another concept is that of the contents or composition which is an area much to the forefront today. Products such as foods or pharmaceuticals must contain the prescribed or guaranteed amounts of the contents desired and must contain none or else not over a permissible amount of undesirable contents. Such requirements are controllable by process and testing controls and are a basic part of the reliability picture.

Then too there is the probability facet. In this imperfect world, there is usually no way to guarantee in absolute terms the functioning of product. About all one can do is to make the probability of functioning sufficiently high.

We may summarize the foregoing with a commonly given definition of reliability.

"The reliability of product is the probability of its successful functioning under prescribed conditions of usage and for the prescribed minimum time or number of cycles". For example, we can let a variable, T, denote the time-to-failure of a given 100 W light bulb. Then the reliability, Say R (t), of the bulb as a function of operating time, t, can be written symbolically as follows:

$$R(t) = P(T>t)$$

where P (T>t) denotes the probability that the variable T exceeds t. In other words, the probability that the bulb does not fail before time t is the reliability of the bulb at time t.

1.2 RELIABILITY FUNCTION:

Suppose that the unit begins to function at the instant $t = 0$ and that a failure occurs at the instant $t = T$. We shall say that T is the lifetime of the unit. Let us suppose that T is a random variable with distribution given by

$$F(t) = P[T < t] \quad \text{------} \quad (1.2.1)$$

The function F (t) is the probability of failure of the unit prior to the instant t. Let us suppose that the function

F(t) is continuous and that there exists a continuous density of probability of failure

$$f(t) = F'(t)$$

These conditions are natural conditions in reliability theory.

Along with this function, we use another function, namely

$$R(t) = 1 - F(t) = P\,[T>t] \text{ ------ } (1.2.2)$$

i.e, the probability of failure-free operation of the unit during the time t. The most common name for this function is "reliability function".

A typical form of the reliability function is shown in Fig.1. This function decreases monotonically, i.e. $R(0) = 1$ and $R(t) \to 0$ as $t \to \infty$.

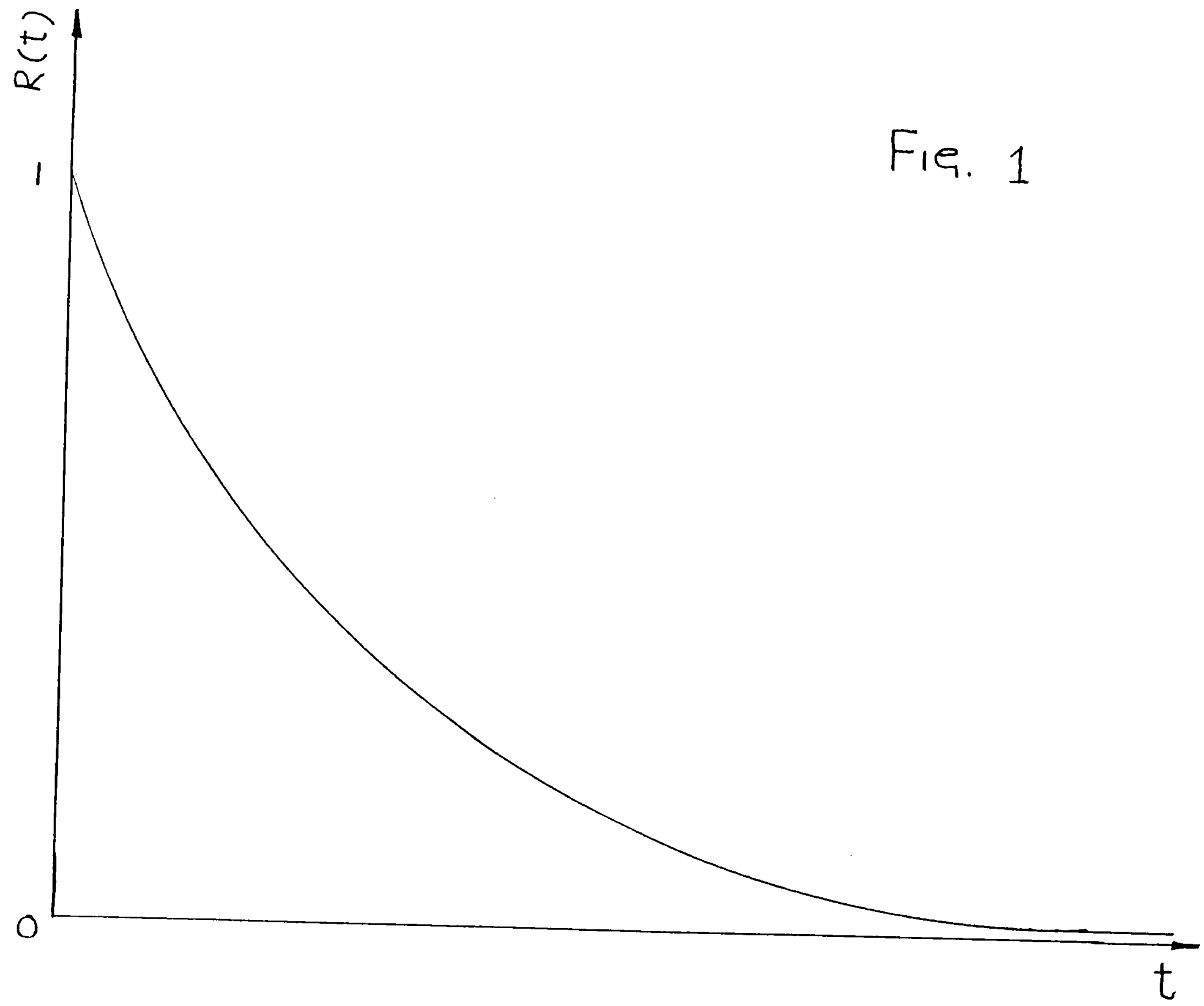
The reliability is often characterized by certain numerical quantities. The most important of these is the mean time of failure free operation, which is defined as the mathematical expectation of the random variable T:

$$T_o = E\,(T) = \int_0^{\infty} t\; f(t)\; dt$$

Integrating by parts

$$T_0 = t\;F(t)\Big|_o^{\infty} - \int_0^{\infty} F\;(t)\;dt$$

Fig. 1



or

$$T_0 = \int_0^{\infty} R(t)\, dt \qquad \text{----} \; (1.2.3)$$

The mean time T_0 is geometrically represented by the area bounded by the coordinate axes and the curve R (t).

Another characteristic of reliability is the variance in the *life length*:

$$V(t) = E(t-T_0)^2 = E(t^2) - T_0^2$$

$$= \int_0^{\infty} t^2 f(t)dt - T_0^2$$

$$= 2\int_0^{\infty} t\, R(t)\, dt - T_0^2 \qquad \text{------} \; (1.2.4)$$

1.3 FAILURE RATE CONCEPT:

The failure rate is the most important and popular characteristic of reliability theory.

A "failure" is any inability of a part or equipment to carry out its specified function.

An "item" may be any part, sub-system, system or equipment which can be individually considered and separately used.

An item can fail in many ways and these failures are classified as follows:

(a) **Causes of failure:**

i) Misuse failure: Failures attributable to the application of stresses beyond the stated capabilities of the item.

ii) Inherent weakness failure: Failures attributable to weakness inherent in the item itself when subjected to stresses within the stated capabilities of that item.

(b) **Times of failure:**

i) Sudden failure: Failures that could not be anticipated by prior examination.

ii) Gradual failure: Failures that could be anticipated by prior examination.

(c) **Degrees of failure:**

i) Partial failure: Failures resulting from deviations in characteristic beyond specified limits not such as to cause complete lack of the required function.

ii) Complete failure: Failure resulting from deviations in characteristic beyond specified limits such as to cause complete lack of the required function.

(d) **Combinations of failures:**

i) Catastrophic: Failures which are both sudden and complete.

ii) Degradation: Failures which are both gradual and Partial.

Thus, there are many Physical causes that individually or collectively may be responsible for the failure of a device at any particular instant. It may not be possible to isolate these physical causes and mathematically account for all of them, and therefore, the choice of a failure distribution is still an art.

Keeping the above difficulties in view, it is of paramount importance to appeal to a concept that makes it possible to distinguish between the different distribution functions on the basis of a physical consideration. Such a concept is based on the failure - rate function, which is known as the HAZARD RATE in reliability. In actuarial statistics the 'hazard rate' goes under the name of 'force of mortality', in extreme-value theory it is called the 'intensity function', and in economics its reciprocal is called 'Mill's Ratio.'

Let $F(t)$ be the distribution function of the time-to-failure random variable T, and let $f(t)$ be its probability density function. Then the hazard rate, $\lambda(t)$, is defined as

$$\lambda(t) = \frac{f(t)}{1 - F(t)} \qquad \text{-----} \quad (1.3.1)$$

Here 1 - F (t) is called the reliability at time t and will be denoted by either R (t). Also

$$\lambda(t) = -R'(t)/R(t), \text{ Since } f(t) = -R'(t) \text{ ------ } (1.3.2)$$

The hazard rate, which is a function of time, has a probabilistic interpretation, namely, $\lambda(t)\,dt$ represents the probability that a device of age t will fail in the interval (t, t + dt), or

$$\lambda(t) = \lim_{\Delta_t \to 0}\left[\frac{P\ [\text{a device of age } t \text{ will fail in the interval } (t, t+\Delta_t)\ |\ \text{it has survived up to } t]}{\Delta t}\right]$$

To assist the choice of $\lambda(t)$, three types of failures generally have been recognized as having a time characteristic. When an equipment is first put into use any inherently weak parts usually fail fairly soon. This is called the 'early failure period' or the 'initial failure'. The early failure rate may, therefore be relatively high, but falls as the weak parts are replaced. There is then a period during which the failure rate is lower and fairly constant. This is called the 'constant failure rate period' or the 'chance failure'. Finally the failure rate rises again as parts start to wear out. This is called the 'Wear-out failure period'.

These three types of failures have been defined by British Standards Institution (B.S.I.) as follows:

Early failure period or Initial failure: That early period,

beginning at some started time and during which the failure rate of some items is decreasing rapidly.

Constant failure rate period or chance failure: That period during which the failure occurs to some items at an approximately uniform rate.

Wear-out failure period : That period during which the failure rate of some items is rapidly increasing due to deterioration processes.

The three types of failures have been classically represented by the bath-tub curve (Fig.2), wherein each of the three segments of the curve represents one of the three time periods: initial, chance, and wear-out.

Given the functional form of $\lambda(t)$, the f (t) and the F (t) could be easily determined. Since

$$\lambda(t) = - R' (t)/R (t)$$

Integrating both sides in the range of (0,t), we have

$$R (t) = \text{Exp} \left[\int_0^t \lambda(s)\, ds \right] \qquad \text{-------} \ (1.3.3)$$

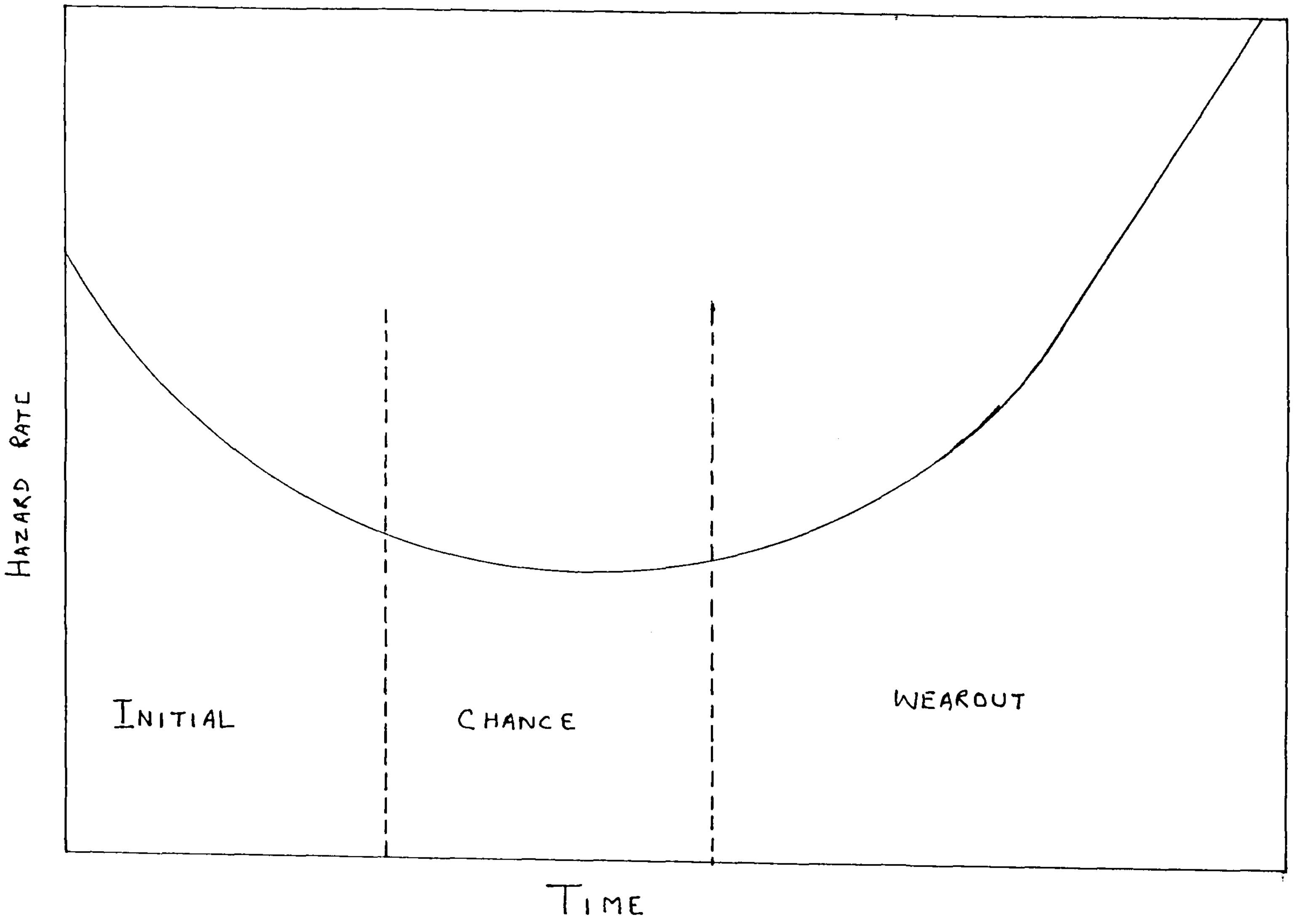


FIG. 2 A TYPICAL (BATHTUB) HAZARD RATE CURVE

Or

$$1 - F(t) = \text{Exp}\left[-\int_o^t \lambda(s)\, ds \right] \quad ----- (1.3.4)$$

Taking derivation, we get

$$f(t) = \lambda(t)\, \text{Exp}\left[-\int_o^t \lambda(s)\, ds \right] \quad ----- (1.3.5)$$

1.4 STATISTICAL FAILURE MODELS

1.4.1 Exponential Failure Model:

The exponential distribution is widely used in reliability. It is inherently associated with the paisson process. Suppose that random 'Shocks' to a device occur according to the postulates of Poisson process. Thus, the random number of shocks X(t) occurring in a time interval of lenght t is described by the poisson distribution

$$P[\, X(t) = n] = \frac{e^{-\lambda t}\, (\lambda t)^n}{n\,!}, \quad n = 0,1,2....\ \lambda, t>0$$

where λ is the rate at which the shocks occur. Further suppose that the device fails immediately upon receiving a single shock and will not fail otherwise. Let the random variable T denote the failure time of the device. Thus

R(t) = P [the device survives at least to time t]

= P [no shocks occur in (0,t)]

$= P[X(t) = 0] = e^{-\lambda t}$

Thus

$$f(t) = -\frac{dR(t)}{dt} = \lambda e^{-\lambda t}, \quad t \geq 0, \ \lambda > 0 \text{ -------- } (1.4.1.1)$$

Which is exponential distribution with parameter λ.

The same expression for the pdf of T could be obtained from the hazard-rate concept. Since the assumption of a random occurrence of shocks with parameter λ implies a constant hazard rate, $\lambda(t) = \lambda$, for $t > 0$. Now f(t) can be obtained from equation (1.3.5) as

$$f(t) = \lambda(t) \exp\left[-\int_0^t \lambda(s)\, ds\right]$$

or,

$$f(t) = \lambda e^{-\lambda t}, \quad t \geq 0, \lambda > 0$$

and eq. (1.3.4) gives

$$F(t) = 1 - e^{-\lambda t}, \quad t \geq 0, \lambda > 0$$

A more general form for the exponential distribution can be obtained if.

$$\lambda(t) = 0, \qquad 0 \leq t < A$$
$$= \lambda, \qquad t \geq A$$

Then,

$$f(t) = \lambda e^{-\lambda(t-A)}, \qquad t \geq A$$
$$= 0, \qquad t < A \qquad \text{--------- } (1.4.1.$$

Often A is referred to as the threshold or the shift parameter.

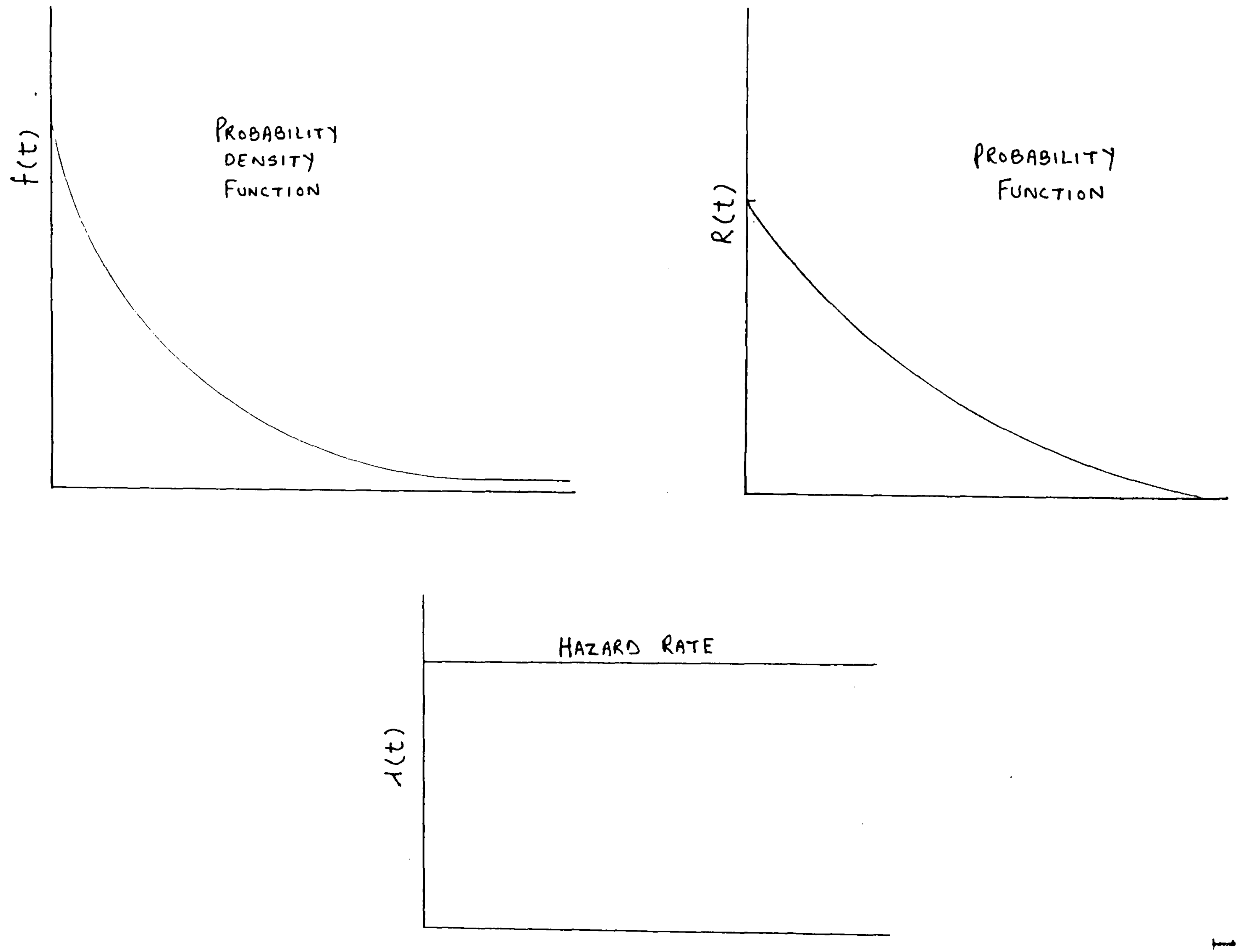


FIG. 3 THE P.D.F, RELIABILITY FUNCTION, AND HAZARD RATE FOR EXPONENTIAL DISTRIBUTION.

1.4.2 The Weibull Failure Law:

The Weibull distribution is widely used in reliability.It was first presented by Weibull (1939) and its use in reliability was discussed by Weibull (1951). The failure law is given by the density function

$$f(t) = \rho\alpha t^{\alpha-1} e^{-\rho t^{\alpha}}, \quad , \rho,\alpha>0, t\geq 0$$

The reliability function is simply

$$R(t) = \exp(-\rho t^{\alpha})$$

and the failure rate function is

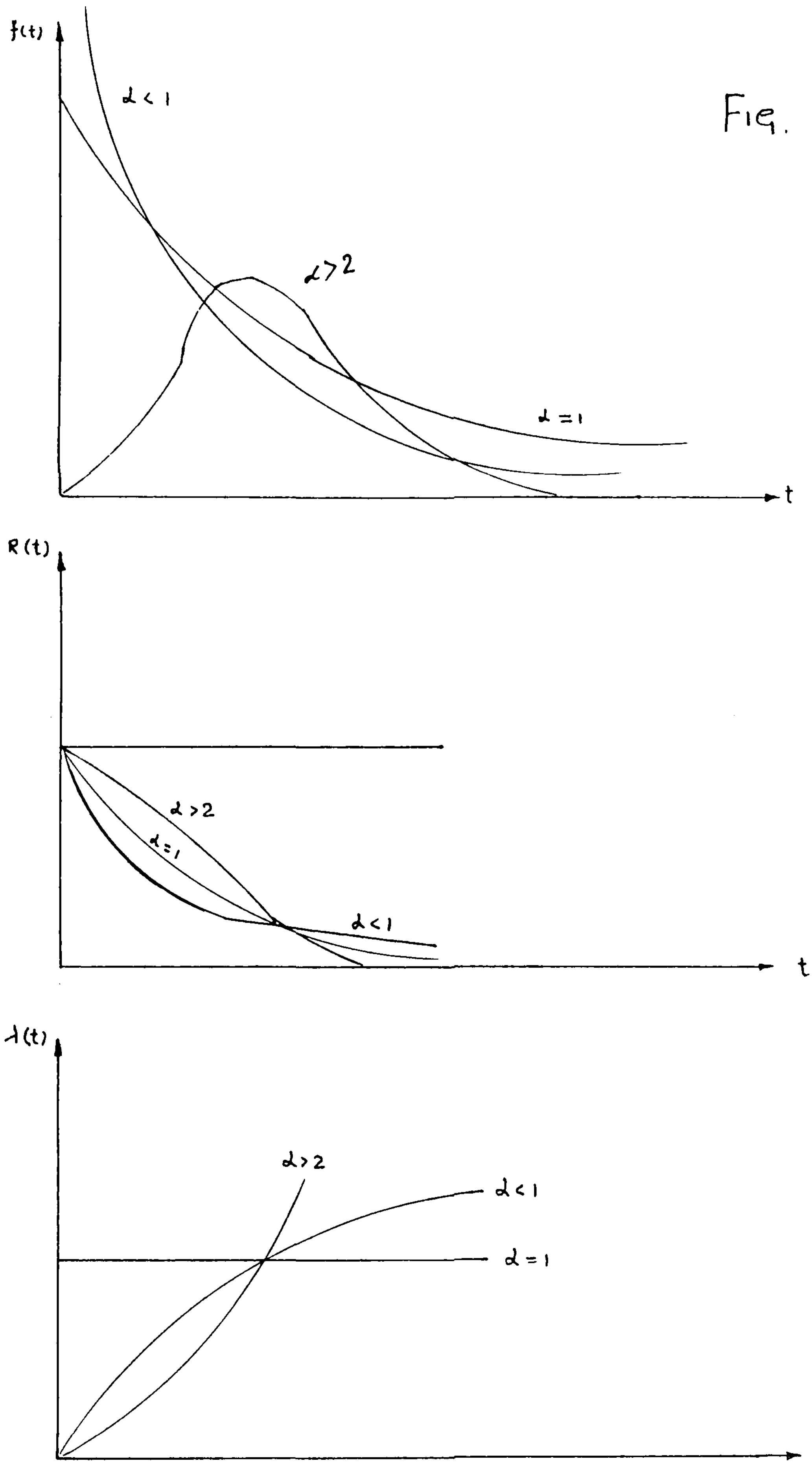
$$\lambda(t) = \rho\alpha t^{\alpha-1}, \rho,\alpha > 0, \; t \geq 0$$

The mean time to failure is calculated as

$$E(T) = \frac{\Gamma(1+1/\alpha)}{\rho^{1/\alpha}} \quad \text{and} \quad V(T) = \frac{1}{\rho^{2/\alpha}}\left[\Gamma(1+2/\alpha) - [\Gamma(1+1/\alpha)]^2\right]$$

For $\alpha = 1$ this is simply the exponential distribution. For $\alpha = 2$ it is referred to as the Rayleighdistribution. The polynomial nature of the failure rate function makes this law particularly useful if, for example, it is desired to approximate failure rate data. These functions are shown for various values of α in Fig.4.

Fig. 4



(i)

(ii)

(iii)

1.4.3 Gamma Failure Model:

The gamma distribution is a natural extention of the exponential distribution and has sometimes been considered as a model in life test problems. It can be derived by considering the time to the k^{th} successive arrival in a Poisson process or, equivalently, by considering the k-fold convolution of an exponential distribution.

Consider a situation in which the unit under consideration operates in an environment where shocks are generated according to a Poisson distribution, with a parameter λ. Further suppose that the unit will fail only if exactly k shocks occur and will not fail until then. One is interested in the random variable $X^{(k)}$, where $X^{(k)}$ denotes the time for the occurrence of the k^{th} shock. In the situation being considered, $X^{(k)}$ represents the time to failure of the unit.

The obtain the pdf of $X^{(k)}$, $f_{X^{(k)}}(x)$, it is to be noted that

$P[X<X^{(k)}<X+\Delta X]$ = P [exactly k-1 shocks occur in (0,X) and exactly 1 shock occurs in $(X,X+\Delta X)$].

Since the number of shocks that occur in [0,X] is given by the Poisson mass function

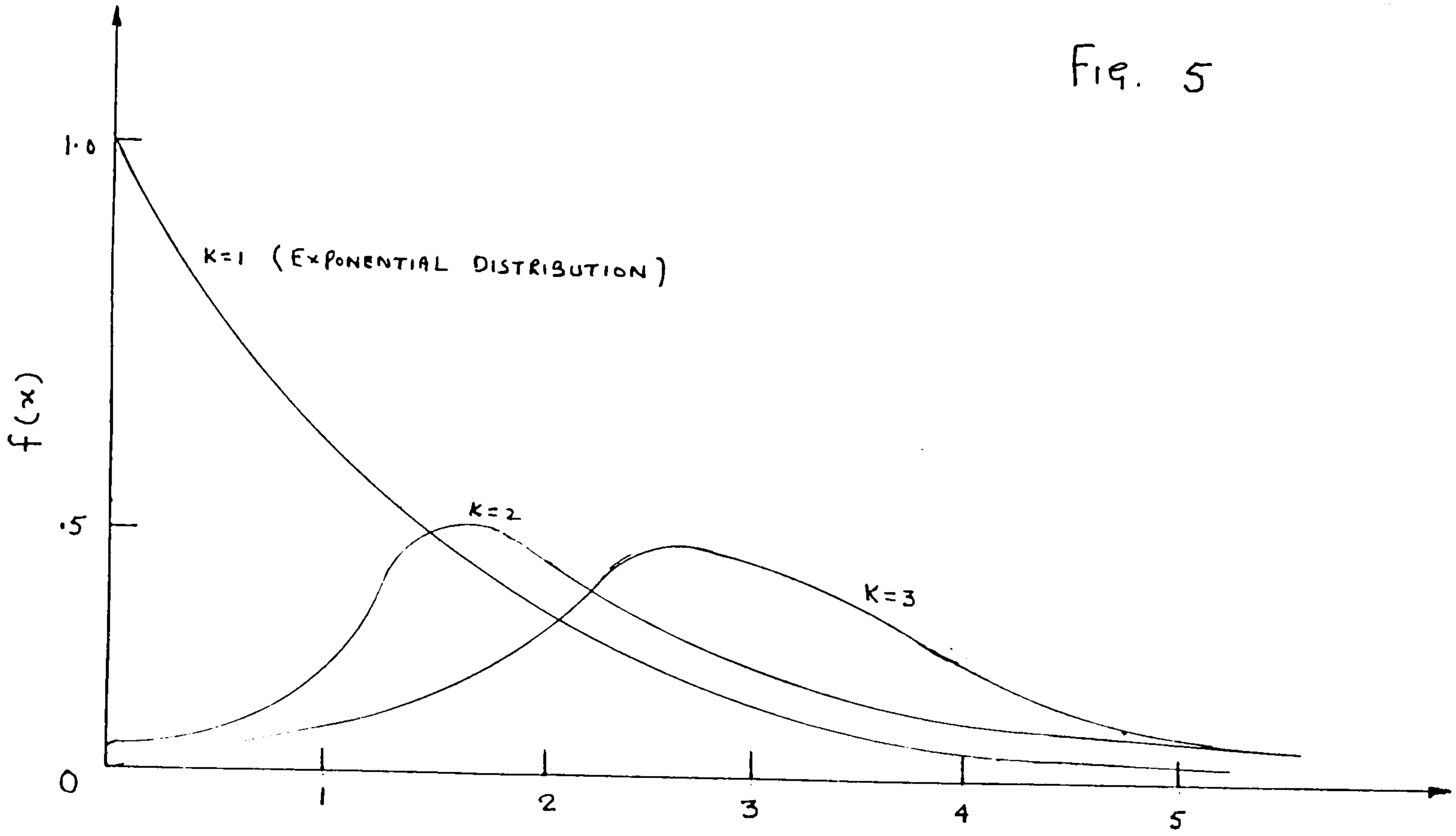


Fig. 5

$$f(X)\Delta X = \lim_{\Delta X\to 0} P(X<X^{(k)}<X+\Delta X) = \frac{e^{-\lambda X}(\lambda X)^{k-1}}{(k-1)!}\lambda\Delta X$$

Hence

$$f_{X^{(k)}}(x) = \frac{e^{-\lambda x}(\lambda x)^{k-1}\lambda}{\Gamma(k)}, \quad k\geq 1, x\geq 0 \quad \text{-------} \quad (1.4.3.1)$$

where $\Gamma(k) = (k-1)!$ is the gamma function.

The distribution function of $X^{(k)}$, $F_{X^{(k)}}(x)$, can be obtained as follows:

$$1 - F_{X^{(k)}}(x) = P(X^{(k)}>x) = P\,[k\text{-}1 \text{ or fewer shocks in } (0,x)$$

$$= \sum_{j=0}^{k-1} \frac{e^{-\lambda x}(\lambda x)^j}{j!}$$

$$F_{X^{(k)}}(x) = \sum_{j=k}^{\infty} \frac{e^{-\lambda x}(\lambda x)^j}{j!} \quad \text{-------} \quad (1.4.3.2)$$

Here $1/\lambda > 0$ is the trivial scale parameter, but $k > 0$, the shape parameter, is essential. For integer values of k, the gamma p.d.f. is also known as the Erlangian probability density function, and, if $k = 1$, the gamma density reduces to an exponential density.

1.4.4 The Normal Law:

Sudden failures of random nature are usually described by an exponential law. On the other hand the failures which arise as a result of wear and tear, of irreverisble physico-chemical changes in the physical parameters of the unit, do not obey the exponential law. These failures are well described by normal law.

Suppose that the reliability of a unit is determined by a single parameter α. Suppose also that the initial value α_0 of the parameters is a normal random variable with small variance.

For a normal reliability law the reliability function is of the form

$$R(t) = \frac{\frac{1}{\sqrt{2\pi}} \int_{t-T_0/\sigma}^{\infty} e^{-t^2/2}\, dt}{\frac{1}{\sqrt{2\pi}} \int_{-T_0/\sigma}^{\infty} e^{-t^2/2}\, dt} \qquad \text{------- (1.4.4.1)}$$

since $\sigma << T_0$, we can write this equation as follows

$$R(t) = \frac{1}{\sqrt{2\pi}} \int_{\frac{t-T_0}{\sigma}}^{\infty} e^{-t^2/2}\, dt \qquad \text{--------- (1.4.4.2)}$$

Where T_0 is the mean life length and $\sigma^2 = V(T)$. The failure rate, $\lambda(t)$, for a normal law is of the following form (Fig.6).

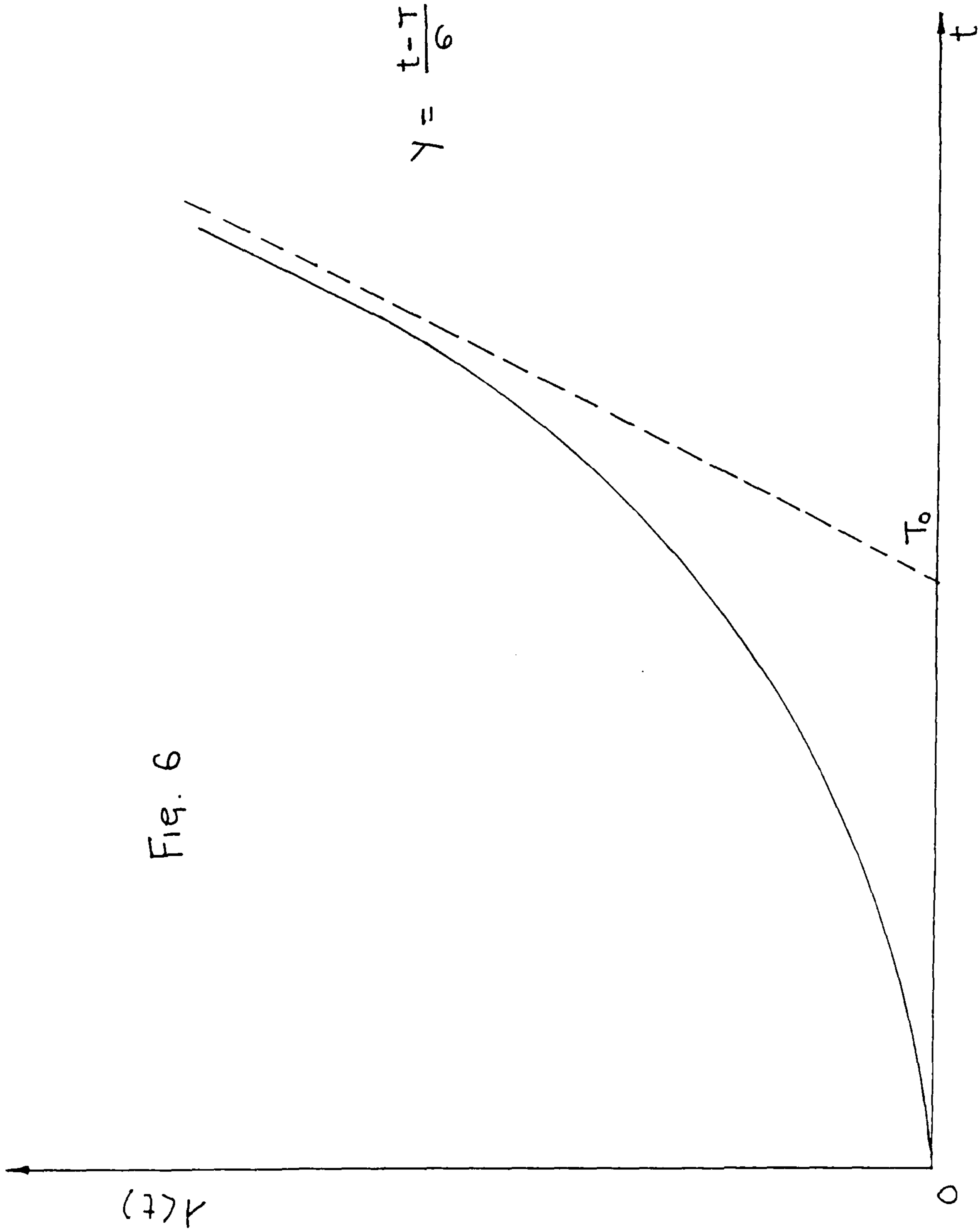


Fig. 6

It increases monotonically and after T_0, it begins to approach as an asymptote, the line $(t-T_0)/\sigma$

1.4.5 The Mukherjee - Islam Model:

The model proposed by Mukherjee and Islam is:

$$f\ (x:\theta,p) = (P/\theta^P)x^{p-1} \quad \theta,p > 0:\ 0 \leq x \leq \theta$$

It is defined by CDF as

$$F(x) = (x/\theta)^P$$

is easily tractable, has a finite range (θ), and includes several important distributions as particular cases. For example, uniform and exponential distributions correspond to $p = 1$ and $p = \infty$, respectively.

It is possible to introduce a location parameter at a time α - a time before which failures cannot occur and, therefore, write the density function as

$$f\ (x) = \frac{p\ (x - \alpha)^{p-1}}{\theta^P}, \alpha \leq x \leq \alpha + \theta,\ \theta,p > 0, \alpha \geq 0$$

It can be easily shown that the asymptotic distribution of the smallest of n observations from this population has the PDF.

$$g(t) = np\,(x/\theta)^{p-1} \exp\,[-n(x/\theta)^{p}]$$

Which is the well - known Weibull density function and arises in the statistical Theory of strength.

Reliability and Failure Rate:

For a mission time X_0, reliability of equipment having this failure time distribution is

$$\overline{F}\,(X_0) = 1 - (X_0/\theta)^{p}$$

The failure rate at time X is

$$r\,(X) = \frac{pX^{p\ -1}}{\theta^{p} - X^{p'}}$$

Since

$$r'(X) = \frac{(\theta^{p} - X^{p})p(p - 1)\ X^{p-2} + p^{2}X^{2}(p-1)}{(\theta^{p} - X^{p})^{2}}$$

the distribution is increasing failure rate (IRF) so long as $p > 1$, When $p < 1$, $r'(X) > 0$ if $X > X_1$ and $r'\,(X) < 0$ if $X < X_1$ where

$$\theta^{p}(p - 1) + X_1^{p} = 0$$

or

$$X_1 = \theta\,(1 - p)^{1/p}$$

This, with $p < 1$, the distribution remains decreasing failure rate (DFR) at least over the first quarter of its life and then the failure rate increases monotonically.

[The fraction $X_1/\theta = (1 - p)^{1/p}$ has the minimum value of 1/4 for $0 < p < 1$.].

The average failure rate over the interval (0,t) is, by definition, $-(1/t)\log[\bar{F}(t)] = -(1/t)\log[1 - (t/\theta)^p]$. Clearly this is increasing in t for all $p > 0$. Thus, the distribution is IFRA though not IFR over the entire range of t.

Given that equipment has survived up to time X, the probability that it will survive until x + y is

$$\bar{F}(x + y)/\bar{F}(x) = \left[1 - \left(\frac{x + y}{\theta}\right)^p\right] \Big/ \left[1 - \left(\frac{x}{\theta}\right)^p\right]$$

For $p > 1$, $\bar{F}(x + y)$ is smaller than $\bar{F}(x)\bar{F}(y) = [1 - (x/\theta)^p][1 - (y/\theta)^p]$ and hence the distribution is new better than used (NBU).

The conditional mean remaining life works out as

$$\int_t^\infty \bar{F}(x)\,dx/\bar{F}(t) = \frac{p}{p + 1}\;\frac{\theta^{p+1} - t^{p+1}}{\theta^p - t^p} - t,$$

which is an increasing function of t. This distribution is new better than used in expectation (NBUE) if

$$(\theta^{p+1} - t^{p+1}) - \theta(\theta^p - t^p) - \frac{p^t}{p+1}(\theta^p - t^p) \leq 0.$$

or if

$$(\theta - t)t^p(p + 1) - p^t(\theta^p - t^p) \leq 0,$$

or if

$$(p + 1)\theta - t^{p+1} - p^{t}\theta^{p} \leq 0.$$

The distribution is otherwise new worse than used in expectation (NWUE).

Estimation of Reliability:

To estimate reliability from a sample of observed failure times, we have to estimate the parameters θ and p. To obtain the maximum-likelihood estimates (MLEs) we note that the likelihood function $L = p^{n}\theta^{-np} \pi x^{p-1}$ is ever decreasing in θ and that $X_{(n)}$, the maximum observation, is the maximum likelihood estimator. For p, the MLE becomes

$$\hat{p} = \frac{1}{\log\theta - \overline{\log x}},$$

where $\overline{\log x}$ = arithmetic mean of log x values. Taking the MLE of θ as $X_{(n)}$ this gives

$$\hat{p} = \frac{1}{\log x_{(n)} - \overline{\log x}}$$

An easier method of estimation is the method of moments. If the first two sample moments $\bar{x}$ and s^2 are equated to μ_1' and μ_2, respectively, p has to be estimated from $p^2v^2 + 2pv^2 - 1 = 0$ where $v = s/\bar{x}$ giving

$$\hat{p} = -1 + (1 + \frac{1}{\sqrt{2}})^{1/2} = -1 + (1 + \frac{\bar{x}^2}{2})^{1/2},$$

and then

$$\hat{\theta} = \frac{\hat{p} + 1}{\hat{p}} \bar{X},$$

1.4.6 Log - Logistic Distribution:

a) Log-logistic cumulative distribution: The population fraction failing by age t is

$$F(t) = -[1 + (t/\theta)^{\delta}]^{-1} \quad , t, \delta, \theta > 0$$

θ is in the same measurement units as t, for example, hour months, cycles, etc. In terms of $\lambda = 1/\theta$

$$F(t) = 1 - [1 + (\lambda t)^{\delta}]^{-1} \quad , t > 0$$

b) log-logistic probability density: It is given by

$$f(t) = \frac{\delta}{\theta} (t/\theta)^{\delta - 1} [1 + (t/\theta)^{\delta}]^{-2}, \quad t > 0$$

Although this model has been used occasionally in life testing applications, it has the advantage (like the weibull and exponential models) of having simple algebric expressions for the survivor and hazard function. It is, therefore, more convenient in handling censored data than the lognormal distribution while providing a good approximation to it except in the extreme tails.

Also,

$$f(t) = \lambda \delta (\lambda t)^{\delta - 1} [1 + (\lambda t)^{\delta}]^{-2} , \quad t > 0$$

c) log-logistic reliability function: The population fraction surviving age t is

$$R(t) = [1 + (t/\theta)^{\delta}]^{-1} , \quad t > 0$$

d) log-logistic hazard function: For a log-logistic distribution

$$h(t) = \frac{\delta/\theta \, (t/\theta)^{\delta - 1}}{1 + (t/\theta)^{\delta}} , \quad t > 0$$

1.4.7 BURR TYPE XII DISTRIBUTION

a) Burr type XII cumulative distribution: The population fraction failing by age t is

$$F(t) = 1 - \frac{1}{[1 + (t/\theta)^{\delta}]^{m}} \quad t > 0$$

The shape parameters δ and m and the scale parameter θ are positive. θ is also called the characteristic life and it has the same units as t. Its failure rate is defined as $\lambda = 1/\theta$. In terms of λ

$$F(t) = 1 - \frac{1}{[1 + (\lambda t)^{\delta}]^{m}} , \quad t > 0$$

b) Burr Type XII probability density: The probability function

$$f(t) = \frac{m(\delta/\theta)(t/\theta)^{\delta-1}}{[1+(t/\theta)^{\delta}]^{m+1}}, \quad t > 0$$

is unimodal if $\delta>1$, and L-shaped if $\delta=1$. For m=1 the Burr type XII distribution is the log-logistic distribution. Also.

$$f(t) = \frac{m(\lambda\delta)(\lambda t)^{\delta-1}}{[1+(\lambda t)^{\delta}]^{m+1}}, \quad t > 0$$

c) Burr type XII reliability function: The population fraction surviving age t is

$$R(t) = \frac{1}{[1+(t/\theta)^{\delta}]^{m}} \quad t > 0$$

d) Burr type XII hazard function: For a Burr type XII distribution, the hazard function is

$$h(t) = \frac{m(\delta/\theta)(t/\theta)^{\delta-1}}{1+(t/\theta)^{\delta}}, \quad t > 0$$

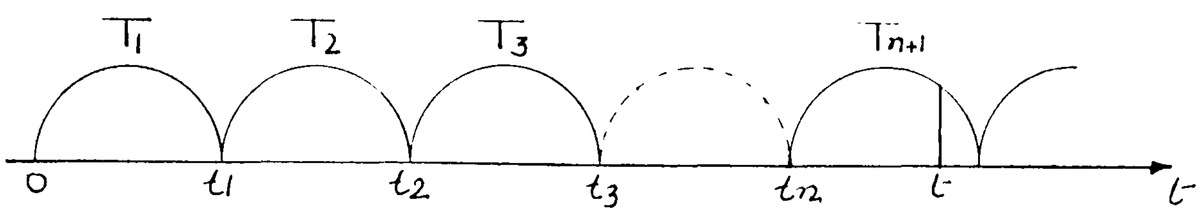
This hazard function is identical to the weibull hazard function aside from the denominator factor $1+(t/\theta)^{\delta}$, it is monotone decreasing from ∞ if $\delta<1$ and is monotone decreasing from $\lambda = 1/\theta$ if $\delta = 1$ if $\delta>1$, the hazard function resembles the lognormal hazard function in that it increases from zero to a maximum at $t = \theta(\delta-1)^{1/\delta}$ and decreases towards zero thereafter.

1.5 RENEWAL PROCESS

A renewal process is a sequence of independent, identically distributed, non-negative random variables, not all 0 with probability 1. Renewal theory plays a significant role in reliability.

In this process we assumed that, after failure, the unit is renewed. This renewal can assume various forms: it can be replaced with a new unit that is identical to it or it can be subjected to maintenance that completely restores all its original properties. We shall assume that as a soon as a unit fails it is renewed instantaneously. Suppose that the unit begins operating at the instant t = 0 and continues operating for a random period of time T_1 and then fails. At that instant, it is replaced with a new unit, which operates for a length of time T_2, then fails and is replaced with a third unit. This process is continued indefinitely. It is natural to assume that the life lengths T_1, T_2 ...... of the units are independent. The random times T_1, T_2 ..... have the same distribution F (t).

$$F(t) = P\ (T_n < t)$$



It is clear from the above figure that the instant of failures or renewals

$t_1 = T_1, t_2 = T_1 + T_2, \ldots. t_n = T_1 + T_2 + \ldots\ldots. T_n, \ldots\ldots$

Constitute a random flow, which we shall call a renewal process.

1.5.1 Renewal function:

Let $\nu(t)$ = number or failures that take place in the time t. Obviously $\nu(t)$ is a random variable. Let us find the distribution of $\nu(t)$. We note that

$$P[\nu(t) \geq n] = P[t_n < t]$$

$$= p[T_1 + T_2 + \ldots.. T_n < t] = F_n(t) \quad \text{---} \quad (1.5.1.1)$$

where the functions $F_n(t)$ are the distribution laws of the t_n and are defined by

$$F_n(t) = \int_0^t F_{n-1}(t - T)\, dF(T), \quad F_1(t) = F(t)$$

Equation (1.5.1.1) implies that

$$R_n(t) = P[\nu(t) = n] = F_n(t) - F_{n+1}(t) \quad \text{-------} \quad (1.5.1.2)$$

In particular,

$$R_0(t) = 1 - F(t).$$

The renewal function H(t), is defined as the mean number of failures that occur up to the instant t. By using equation (1.5.1.2), we get

$$H(t) = E\,[\nu(t)] = \sum_{n=1}^{\infty} n\, R_n(t)$$

$$= \sum_{n=1}^{\infty} n\, F_n(t) - \sum_{n=2}^{\infty} (n-1)\, F_n(t)$$

$$= \sum_{n=1}^{\infty} F_n(t) \qquad \text{----------} \quad (1.5.1.3)$$

Also, $h(t) = H'(t)$

The function $h(t)$ is called the renewal density. It is equal to the mean number of failures that take place in a unit interval begining at the instant t. From eqn. (1.5.1.3), we have

$$h(t) = \sum_{n=1}^{\infty} f_n(t) \qquad \text{where} \quad f_n(t) = F_n'(t)$$

1.6 EXPERIMENTAL PLANS AND LIFE TEST PROCEDURES

In some situations, physical constraints related to the problem under study, or a lack of prior knowledge about the problem, can make precise planning of an investigation difficult. In well-controlled situations, on the other hand, experiments can often be planned to satisfy defined objectives. Much of the discussion concerns life test procedures, for several reasons. One is that life test plans with stated economic objectives are important in many areas and widely used. A second reason is that many of the considerations involved with them are relevant in planning any life time distribution investigation. Finally, by examining different experimental plans for the relatively simple exponential model. we gain insight into the difficulties of designing plans for other distributions.

The most common life testing problem involves testing a specific value θ_o of θ against values less than θ_o. For example, a consumer may want the mean lifetime of a particular type of item to be satisfactorily high. With this in mind, a plan is set up whereby one can test, that the mean lifetime is θ_o, against the alternative that it is less than θ_o. We, therefore, consider testing.

$$H_o: \theta = \theta_o \text{ vs } H_1: \theta < \theta_o \quad \text{----------} \quad (1.6.0)$$

Life tests plans are generally designed so that the size and power of the test at some particular value $\theta_1 < \theta_o$ are specified. The size of the test is defined as

$$\alpha = P(\text{reject } H_o: \theta = \theta_o$$

and the power function, defined for $\theta_1 < \theta_o$, is given by

$$P(\theta_1) = P(\text{regect } H_o: \theta = \theta_1)$$

1.6.1 TYPE II CENSORED (NON REPLACEMENT) LIFE TEST PLANS

Consider the problem of testing the hypothesis (1.6.0) on the basis of a type II censored sample containing the r smallest lifetimes $t_{(1)} < \dots\dots < t_{(r)}$ in a total sample of size n. For a given r and n, a size uniformly most powerful test of H_o versus H_1 exists and has acceptance rule of the form

$$\text{Accept } H_o \text{ if } \hat{\theta} > C_\alpha = \frac{\theta_o \chi^2_{(2r),\alpha}}{2r} \qquad \text{------ (1.6.1.1)}$$

where

$$\hat{\theta} = [\ \sum t_{(i)} + (n-r)\ t_{(r)}]/r$$

For any positive integer r one can get a size α test. If we also require the power of the test at $\theta = \theta_1$ to be $1 - \beta$, then

$$P(\theta_1) = P\ (\hat{\theta} \le C_\alpha\ ;\ \theta = \theta_1) = 1 - \beta$$

But if $\theta = \theta_1$ then $2r\hat{\theta}/\theta_1 \sim \chi^2_{(2r)}$ and so

$$P(\theta_1) = P\ \left(\frac{2r\hat{\theta}}{\theta_1} \le \frac{2rC_\alpha}{\theta_1} \right)$$

$$= P\left(\chi^2_{(2r)} \le \frac{2rC_\alpha}{\theta_1} \right) \qquad \text{----------- (1.6.1.2)}$$

Thus $\chi^2_{(2r),1-\beta} = 2rC_\alpha / \theta_1$ or since $C_\alpha = \theta_o \chi^2_{(2r),\alpha} / 2r$

$$\frac{\chi^2_{(2r),\alpha}}{\chi^2_{(2r),1-\beta}} = \frac{\theta_1}{\theta_o} \qquad \text{------- (1.6.1.3)}$$

Hence to make $P(\theta_1)$ equal to $1-\beta_1$ we must choose r such that (1.6.1.3) is satisfied.

1.6.2 SOME OTHER LIFE TEST PLANS

There are many ways to run a life test experiment. Other possibilities include plans with Type I censoring, a mixture of Type - I and Type - II censoring or a sequential procedure. In addition, tests can sometimes be run with replacement, whereby items that fail are immediately replaced by new items, so that there are alwasy n items on test. Still another possibility is to use partial replacement, replacing only a portion of the failed items. A few plans are given below.

1.6.2.1 TYPE II CENSORING WITH REPLACEMENT

Sometimes it is feasible to replace failed items immediately, with the result that n items are continually on test. If the test is terminated at the time T_r, of the rth item failure, then there is Type II censoring with replacement. The likelihood function is

$$L(\theta) = \frac{1}{\theta^r} e^{-\sum t_i/\theta}$$

where Σ_{t_i} is the total observed lifetime, or the "total time on test". Since there are n items on test at all times and the test terminates at time T_r, Σ_{t_i} must equal nT_r, and T_r is sufficient for θ.

1.6.2.2 TYPE I CENSORING WITH REPLACEMENT

If failed items are replaced immediately, so that n items are always on test, and if testing terminates at some prespesified time L_o, then there is Type I censoring with replacement. The likelihood function is

$$L(\theta) = \frac{1}{\theta^r} e^{-\Sigma t_i/\theta}$$

where r is the observed number of failures and Σ_{t_i} is the total time on test.

1.6.2.3 TYPE I CENSORING WITHOUT REPLACEMENT

If each device that fails is not replaced by a new one, and if test is terminated after a prespecified number of failures have occurred. In a type I censored test the test length is specified to be some fixed number L_o. The likelihood function is

$$L(\theta) = \frac{1}{\theta^r} \exp\left(- \sum_{i=1}^{r} \frac{t_{(i)}}{\theta} \frac{(n-r)L_o}{\theta} \right)$$

1.6.2.4 SEQUENTIAL PLANS

Epstein and Sobel (1955) present a test based on Wald's sequential probability ratio test, in which the decision made at time t essentially depends on the inequality

$$B < \left(\frac{\theta_o}{\theta_1}\right)^{r(t)} \exp\left[(\theta_1^{-1} - \theta_o^{-1})\, T(t)\right] < A \quad \text{----} \quad (1.6.2A)$$

where $r(t)$ is the number of failures observed by time t and $T(t)$ is the total time on test up to time t, that is, the total lifetime lived by all items, failed and unfailed, up to time t. At time t experimentation continues as long as (1.6.2A) is satisfied. On the other hand, if the function in the middle of (1.6.2.1) is $\leq$ B, H_o is rejected, and if it is $\geq$ A, H_o is accepted.

1.7 ACCELERATED LIFE TESTS

Many devices such as electronic items have very high reliability when operating within their intended normal use environment. This presents problems in measuring the reliability of such devices because a very long period of testing under the actual operating conditions would be required to obtain sufficient data to estimate the reliability. Even if this testing could be accomplished, the time frame is such that the devices may become obsolute before their reliability is established due to the high rate of technological advances. Also, it would be difficult to conduct the testing in laboratory.

One solution to the problem of obtaining meaningful life test data for high reliability devices is accelerated life testing. This type of testing involves observing the performance of these kinds of devices operating at higher stress levels than usual to obtain failures more quickly. In order to shorten product life, it is a well established engineering practice to use certain stresses or accelerating variables, such as higher levels of temperature, voltage, pressure, vibration, etc., than the normal operating level.

The main difficulty of accelerated life testing lies in using the failure data obtained at the accelerated, or higher stress, conditions to predict the reliability, mean life, or other quantities under the normal use condition. Extrapolation from the accelerated stresses to the normal use stress is done by choosing an appropriate model, called an acceleration model. The choice of an acceleration model calls for a knowledge of the variation of failure behaviour with environment. In parametric method, this involves functional relationship between the parameters of the failure distributions and the environmental stresses. The relationship may also involve unknown parameters. In nonparametric approaches, where no specific form of the failure distribution is specified, the change in the failure distribution due to a change in environmental stress is assumed. In either the parametric or nonparametric, all unknown parameters must be estimated from the accelerated test data in order to extrapolate to the normal use stress.

Four acceleration models are used, i.e. power rule model the Arrhenius model, the Eyring model, and the generalized erying model. These models will be discussed by Mann, Schafer, and Singhpurwall (1974).

1.7.1 ACCELERATION MODELS

The use of accelerated life testing to make inferences about the normal use life distribution requires a model to relate the life length to the stress levels that are to be applied to the items being tested. This model is referred to as the acceleration model.

Here some acceleration models that have been used in parametric and nonparametric method will be described briefly.

In parametric, suppose the life time random variable X_i^0 of items in an environment described by a constant stress level V_i has a probability distribution $F^o(t;\underline{\theta}_i)$

depending on a vector of parameter $\underline{\theta}_i$. Two assumptions which are made (Mann, Schafer, and Singpurwalla, 1974) are

i) The change in stress level does not change the type of the life time distribution $F^0(t;\underline{\theta})$, but changes only the parameter values.

ii) The relationship between the stress level V and the parameters $\underline{\theta}$. say $\underline{\theta}$ = m (V; α, β...), is known except for one or more of the acceleration parameter α, β..., and that the relationship is valid for a certain range of the elements of V. The objective here is to obtain estimates of the parameters α, β... based on life test data obtained at large values of V and make inferences about $\underline{\theta}$ for the normal use stress V_o.

The exponential distribution with parameter λ is widely used as a lifetime distribution. So the acceleration models will be discussed here for exponential distributions. Several authors have considered other lifetime distributions such as weibull (Mann, 1972, and Nelsen, 1975), extreme value (Meeker and Nelson, 1975, and Nelsen and Meeker, 1978), and lognormal (Nelson and Kielpinski, 1976), for example. Suppose that under constant application of single stress at level V_i, the item being tested has an exponential lifetime distribution with mean μ_i given by

$$f^{o}(t;\lambda_i) = \lambda_i e^{-\lambda_i t}, \quad t \geq 0, \; i > 0$$
$$= 0, \text{ otherwise.}$$

Then $\mu_i = 1/\lambda_i$ is the mean time to failure under stress level V_i. The following acceleration models (relationships between λ_i and V_i) have been suggested in the literature.

i) The Power Rule (or inverse power) Model:

This model can be derived by considerations of kinetic theory and activation energy. This model has applications to fatigue testing of metals, the dielectric breakdown of capacitors, and aging of multicomponent systems. The model is

$$\mu_i = \alpha V_i^{-\beta}, \qquad > 0, \; \beta > 0$$

and this implies that the mean time of failure μ, decreases as the β^{th} power of the applied voltage V. It is desirable to estimate α and β from life test data at stress levels $V_1,\ldots V_k$ and make inferences about $\mu_o = 1/\lambda_o$ at the normal use stress V_o.

ii) The Arrhenius Model:

This model expresses the degradation rate of a parameter of the device as a function of its operating temperature. It is usually applied to thermal aging and is applicable to semiconductor malerials. Here

$$\lambda_i = \text{Exp}\ (\alpha - \beta/V_i)$$

is the model, where V_i denotes the temperature stress and α and β are unknown parameters to be estimated in order to make inferences about λ_o at normal temperature level V_o.

iii) The Eyring Model for a Single stress:

This model can be derived from principles of quantum mechanics and its expresses the time rate of degradation of some device parameter as a function of the operating temperature. Here

$$\lambda_i = V_i \ \exp\ (\alpha - \beta/V_i)$$

is the model.

iv) The Generalized Eyring Model:

This model has application to accelerated testing of devices subjected to a constant application of two types of stresses, one thermal and one nonthermal. The model is

$$\lambda_i = \alpha T_i \ \exp\ (-\beta/K\,T_i)\ \exp\ (\gamma V_i + \delta\, V_i/KT_i)$$

where α,β,τ and are unknown parameters to be estimated, K denotes Boltzmann's constant, whose value is 1.38×10^{-16} erg/degree kelvin, and T_i is thermal stress level and V_i is the nonthermal stress. In the absence of a nonthermal stress, this model reduces to

$$\lambda_i = \alpha T_i \exp(-\beta / T_i).$$

Chernoff (1962) considered an acceleration model for exponential life time with mean $\mu_i = (\alpha V_i + \beta v_i^2)^{-1}$ where $\alpha > 0$ and $\beta > 0$ were unknown parameters. Thus λ_i was a quadratic function of the stress level. Chernoff also considered models for three dimensional vector stresses

$$\underline{V}_i = (V_{1i}, V_{2i}, V_{3i}).$$

In partially nonparametric approaches to inference from accelerated life tests, no particular form of the life time distribution is assumed, but an acceleration model is used (see, for example, Shaked, Zimmer, and Ball, 1979, Selhuraman and Singpurwalla, 1982, Shaked and Singhpurwalla, 1982, Basu and Ebrahimi, 1982, and Shaked and Singpurwalla, 1983). Shaked, Zimmer, and Ball (1979) assumed that the K accelerated stress levels V_1, ... V_k were selected of stresses V_i, V_j, $i, j = 0, 1.., K$, a known function m existed. Therefore the life time distributions satisfied.

$$F_{v_j}(t) = F_{v_i}[m(\ , V_j, V_i, t)], \quad t > 0,$$

where α is an unknown parameter, the form of F_{v_i} is not assumed to be known. Various choices of m gives the power rule, Arrhenius, Eyring, etc., acceleration models. The other references assume models for special cases of m.

In a totally nonparametric setting, there is no assumption made about the form of the lifetime distribution at the various stress levels nor about the forms of an acceleration model. In this setting, the life distributions are stochastically ordered with respect to increasing levels of stress (Barlow and Schever, 1971) or that the lifetime distribution at two distinct stress levels differ only by a scale change. For these procedures, it must be assumed that failure data are available from the normal use stress as well as from accelerated stresses.

The design aspects of accelerated life testing experiments involve the selection of stress levels. The number of stress levels, and the number of items to be tested at each stress level. A null-designed estimaters allow for censoring.

CHAPTER - II

DEVELOPMENT OF ALT SAMPLING PLANS FOR EXPONENTIAL DISTRIBUTIONS

2.1 YUM & KIM (1990) ALT PLAN:

INTRODUCTION

Several life-test acceptance procedures have been developed, in the past, assuming that life tests are conducted at the use condition. Since high reliabilities are usually specified for the latest equipments, so such acceptance procedures may prove impractical in terms of the amount of time required. To overcome the problem it is better to introduce "acceleration in time" to life tests.

We want to develop an acceptance Sampling plan for testing the hypothesis H_0: $\theta_u = \theta_o$, where θ_u is the mean of an exponential lifetime distribution at the use condition, against the alternative H_1: $\theta_u = \theta_1 < \theta_o$ subject to the conditions that the producers and consumer's risks are met. We assume that

a) Life tests are conducted at two overstress levels and

b) as soon as a certain number of failures are observed the test at each overstress level is terminated (i.e., Type II censoring).

The distribution of the test statistic 'W' which is a quotient of two independent random variables each of which is a rational power of a chi-square random variable is characterised by the H-function discussed by carter and springer (1977) or springer (1979).

2.2 THE ALT MODEL AND THE MAXIMUM LIKELIHOOD ESTIMATOR OF θ_u

Let the lifetime T of a test unit at stress level S has p.d.f.:

$$f(t) = \theta^{-1}\exp(-t/\theta), \quad t>0$$

Let

$$\theta = \exp(\beta_o + \beta_1 S) \qquad \text{---------} (2.2.1)$$

Where β_o and β_1 are unknown constants

The relation (2.2.1) finds maximum use in ALT. It includes the inverse power model and Arrhenius reaction rate model for single stress.

In the proposed ALT, two overstress levels are introduced.

a) the two stress level S_l is chosen such that $S_u<S_l<S_h$ where S_u is the use stress level and S_h is the high stress level.

b) the high stress level S_h is assumed to be prespecified.

Without loss of generality, the high stress level and the use stress level are standardized to be 1 and 0 respectively. Following transformation helps us to achieve this

$$S = (s' - S'_u)/(S'_h - S'_u)$$

Where a prime is used to indicate the original scale of the Stress.

In the proposed ALT, under the constant application of stress S_i we put on test n_i $(i = l,h)$ units and as soon as r_i failures are observed the life test at each overstress level is terminated. It is assumed that the lifetimes of test units are independent. It depends on the policy adopted to either replace the failed units or retain them.

The unknown constants β_o and β_1 of the relation(2.2.1) can be estimated with the help of the results of the above life tests to make inferences on the mean of an exponential lifetime distribution at the use condition (θ_u).

We prove following lemma to show how the mean lifetime at the use condition is related to those at the low and high stress levels under the stated model.

LEMMA

If the relationship between θ and S is given by(2.2.1) and if stress levels are standardized such that $S_u = o$ and $S_h = 1$, then

$$\theta_u = \theta_l^p \; / \; \theta_h^q \qquad \text{------------} \; (2.2.2.)$$

Where

$$P = 1/(1-S_l),$$

$$q = S_l/(1-S_l)$$

Proof

Since the stress levels are standardized we obtain the following results from relationship (2.2.1)

$$\theta_u = \exp\,(\beta_o) \qquad \text{------------} \; (2.2.3)$$

$$\theta_l = \exp\,(\beta_o + \beta_1 S_l) \qquad \text{------------} \; (2.2.4)$$

$$\theta_h = \exp\,(\beta_o + \beta_1) \qquad \text{------------} \; (2.2.5)$$

From equations (2.2.4) & (2.2.5)

$$\ln \theta_l = \beta_o + \beta_1 \; S_l \qquad \text{-------------} \; (2.2.6)$$

$$\ln \theta_h = \beta_o + \beta_1 \qquad \text{-------------} \; (2.2.7)$$

$$\beta_o = \ln \theta_h - \beta_1$$

Substituting value of β_o in Eq. (2.2.6)

$$\ln \theta_l = \ln \theta_h - \beta_1 + \beta_1 S_l$$

$$\ln \theta_l - \ln \theta_h = \beta_1 \; (S_l - 1)$$

or $$\ln \theta_h - \ln \theta_l = \beta_1 \; (1 - S_l)$$

$$\ln \left(\frac{\theta_h}{\theta_l}\right) = \beta_1 \; \left(\frac{1}{P}\right) \qquad \because P = \frac{1}{1-S_l}$$

$$P \ln (\theta_h/\theta_l) = \beta_1$$

$$\ln (\theta_h/\theta_l)^p = \beta_1$$

or $\beta_1 = \ln (\theta_h/\theta_l)^p$

Substituting value of β_1 in Eq. (2.2.7)

$$\ln \theta_h = \beta_0 + \ln (\theta_h/\theta_l)^p$$

$$\beta_0 = \ln \theta_h - \ln (\theta_h/\theta_l)^p$$

$$= \ln [\ \theta_h | (\theta_h/\theta_l)^p]$$

$$= \ln [\ \theta_l^p \ / \theta_h^{p-1}\ ]$$

or $\beta_o = \ln [\theta_l^p \ / \theta_h^q\ ]$ $\quad p - 1 = \dfrac{1}{1 - S_l} - 1$

$$= \frac{S_l}{1 - S_l} = q$$

From eq. (2.2.3)

$$\theta_u = \exp (\beta_o)$$

$$\ln \theta_u = \beta_o$$

$$\Rightarrow \quad \theta_u = \theta_l^p \ / \ \theta_h^q$$

We now show that inferences on θ_u may be directly based upon $\hat{\theta}_i$ (an estimator of the mean lifetime at S_i, $i=l,h$) and hence estimates of β_0 and B_1 need not be calculated.

At stress level S_i, the MLE of θ_i is given by (Epstein and Sobel, 1953).

$$\hat{\theta}_i = \begin{cases} \dfrac{\sum_{j=1}^{r_i} t_{ij} + (n_i - r_i)\, t_{i,r_i}}{r_i} & \text{(without replacement)} \\[2ex] \dfrac{n_i\, t_{i,r_i}}{r_i} & \text{(with replacement)} \end{cases} \qquad \text{----- (2.2.8)}$$

Where t_{ij}, $i=l,h$ and $j = 1,2,\ldots\ldots, r_i$, denotes the failure time of the jth unit at stress level S_i. It is well known that $2r_i\hat{\theta}_i/\theta_i$ has a chi-square distribution with degrees of freedom $2r_i$ (Epstein and sobel, 1953).

From the lemma, the ML estimator of θ_u is given by

$$\hat{\theta}_u = \hat{\theta}_l^p \,/\, \hat{\theta}_h^q \qquad \text{---------------- (2.2.9)}$$

2.3 ACCEPTANCE PROCEDURE

Our aim is to test $H_o : \theta_u = \theta_o$ against $H_1 : \theta_u = \theta_1 < \theta_o$. The proposed acceptance rule is to accept H_o if

$$\hat{\theta}_u \geqslant C.$$

Let α be the size of the test. Then,

$$\Pr\left[\hat{\theta}_u = \hat{\theta}_l^p / \hat{\theta}_h^q \geqslant C / \theta_u = \theta_o\right] = 1 - \alpha ,$$

Which can be rewritten as

$$\Pr\left[W \geqslant \frac{(2r_l)^p C}{(2r_h)^q \theta_o}\right] = 1 - \alpha , \quad \text{------} (2.3.1)$$

Where

$$W = \frac{(2r_l \hat{\theta}_l / \theta_l)^p}{(2r_h \hat{\theta}_h / \theta_h)^q} \quad \text{------} (2.3.2)$$

Note that W is a quotient of two independent random variables, each of which is a rational power of a Chi-square random variable. Furthermore, W depends only on r_l, r_h and S_l.

From (2.3.1)

$$\frac{(2r_l)^p C}{(2r_h)^q \theta_o} = w_\alpha , \quad \text{-------} (2.3.3)$$

or equivalently

$$C = \frac{(2r_h)^q}{(2r_l)^p} \theta_o w_\alpha , \quad \text{--------} (2.3.4)$$

where w_α is the α-th quantile of the distribution of W. In addition to the size of the test, we also require the power at $\theta_u = \theta_1$ to be $1-\beta$. That is,

$$\text{Power } (\theta_1) = \Pr[\hat{\theta}_u = \hat{\theta}_\ell^p / \hat{\theta}_h^q < C/\theta_u = \theta_1]$$

$$= 1 - \beta,$$

Which can be also rewritten as

$$\Pr\left[W < \frac{(2r_\ell)^p C}{(2r_h)^q \theta_1}\right] = 1 - \beta$$

Therefore, we must have

$$\frac{(2r_\ell)^p C}{(2r_h)^q \theta_1} = w_{1-\beta} \qquad \text{--------- } (2.3.5)$$

From (2.3.3) and (2.3.4) we have to choose r_ℓ, r_h and S_ℓ such that the following holds.

$$\frac{w_{1-\beta}}{w_\alpha} = \frac{\theta_o}{\theta_1} \qquad \text{---------} (2.3.6)$$

For a given S_ℓ, there may not exist integral values of r_ℓ and r_h such that (2.3.6) is exactly satisfied. One way of avoiding this difficulty is to allow,
Power (θ_1) to be greater than or equal to $1 - \beta$, and then select "smallest" integral values of r_ℓ and r_h such that the left hand side of (2.3.6) is greater than or equal to θ_o/θ_1.

2.4 DISTRIBUTION OF W

$$W = \frac{(2r_{\ell}\hat{\theta}_{\ell}/\ \theta_{\ell})^{p}}{(2r_{h}\hat{\theta}_{h}/\theta_{h})^{q}}$$

This test statistic W is a quotient of two independent random variables each of which is a rational power of a chi-square random variable.

The distribution of W can be characterized by the H-function discussed by carter and springer (1977) or springer (1979).

2.5 HUI K. HSIEH (1994) ALT PLAN

INTRODUCTION

In order to shorten the experimental time, most laboratory experiments on product lifetimes are conducted under higher - than - normal - use stress levels. This is known as an accelerated life test (ALT). To make inferences on parameters associated with the lifetimes under normal use stress, we make use of the lifetimes obtained from the test.

Here we obtain an alternative formulae for computing the type I and type II error probabilities. The new formulae involves only single integral and an explicit formula for assessing the numerical error due to truncation of each improper integral is also obtained.

2.6 THE PROBLEM AND NOTATION

Assume that the lifetime T of a test unit at stress level S has an exponential distribution the p.d.f of which is given by

$$f(t) = (1/\theta) \exp(-t/\theta) \quad \text{----------- (2.6.1)}$$

Where $\theta = \exp(\beta_o + \beta_1 S)$, ----------- (2.6.2)

and β_o and β_1 are unknown constants

Let S_u be the normal use stress level,

and Let S_l and S_h be two higher - than - normal - use stress levels ($S_u < S_l < S_h$). Their θ values are

$$\left.\begin{aligned} \theta_u &= \exp(\beta_o + \beta_1 S_u) \\ \theta_l &= \exp(\beta_o + \beta_1 S_l) \\ \theta_h &= \exp(\beta_o + \beta_1 S_h) \end{aligned}\right| \quad \text{-------------- (2.6.3)}$$

respectively.

In section 2.2 we obtained

$$\theta_u = (\theta_l)^P / (\theta_h)^q \quad \text{-------------- (2.6.4)}$$

Where $P = 1/(1-S_l^*)$, $q = S_l^*/(1-S_l^*)$, and

$S_j^* = (S_j - S_u)/(s_h - S_u)$ for $j = u, l, h$.

Note that $\overset{*}{S}_u = 0$, $\overset{*}{S}_h = 1$ and $0<\overset{*}{S}_\ell<1$

The Hypotheses are:

$$\left.\begin{array}{ll} & H_o: \theta_u = \theta_o, \\ \text{and} & H_1: \theta_u = \theta_1 \end{array}\right| \quad \text{-----------} \quad (2.6.5)$$

where θ_o and θ_1 are two given numbers, $0<\theta_1<\theta_o$. We want to develop acceptance sampling plans for an ALT Subject to specified producer's risk α and consumer's risk β. Here, α and β correspond to the type I and type II error probabilities, respectively.

Data will be obtained from a life test conducted under the two high stress levels, S_ℓ and S_h, and under a type II censoring plan without replacement.

For each i (i=l,h), a sample of n_i items is put on test at level S_i, and the experiment is terminated as soon as r_i failures are observed.

Let $t_{i1}, \ldots\ldots, t_{ir_i}$ be the first r_i ordered failure times. The MLE of θ_u is then

$$\hat{\theta}_u = (\hat{\theta}_\ell)^p/(\hat{\theta}_h)^q \quad \text{-------------} \quad (2.6.6)$$

Where

$$\hat{\theta}_i = \frac{\sum_{j=1}^{r_i} t_{ij} + (n_i - r_i)\, t_{ir_i}}{r_i}, \quad i=\ell,h \quad \text{---}(2.6.7)$$

and $\hat{\theta}_l$ & $\hat{\theta}_h$ are assumed to be statistically independent.

To test H_o: $\theta_u = \theta_o$ against H_1: $\theta_u = \theta_1 < \theta_o$. The proposed acceptance rule is to accept H_o if $\hat{\theta}_u > C$ for some constant C and to reject H_1 otherwise. Let

$$W = (2r_l\hat{\theta}_l/\theta_l)^p/(2r_h\hat{\theta}_h/\theta_h)^q \qquad \text{-----------} \quad (2.6.8)$$

Test statistic W is a quotient of two independent random variables each of which is a rational power of a Chi-square random variable.

For a specified $(S_l, S_h, \theta_o, \theta_1, \alpha, \beta)$ the censoring numbers (r_l, r_h) and the constant C will be chosen to satisfy.

$$1-\alpha = \Pr[\hat{\theta}_u = \hat{\theta}_l^p / \hat{\theta}_h^q \geq C \mid \theta_u = \theta_o]$$

Which can be rewritten as

$$1-\alpha = \Pr[W \geq (2r_l)^p C/(2r_h)^q \theta_o]$$

or

$$\alpha = \Pr[W \leq \frac{(2r_l)^p C_o}{(2r_h)^q}] \qquad \text{--------} \qquad (2.6.9)$$

Where $C_o = C/\theta_o$

Also,
$$1-\beta \leq \Pr[\hat{\theta}_u \leq C \mid \theta_u = \theta_1]$$
$$= \Pr[W \leq \frac{(2r_l)^p C}{(2r_h)^q \theta_1}]$$

$$= \Pr \left[W \leqslant \frac{(2r_{\ell})^{p}\, C_o}{(2r_h)^{q}\, \phi} \right] \quad \text{-------} \quad (2.6.10)$$

Where $C_o = \dfrac{C}{\theta_o}$, $\phi = \dfrac{\theta_1}{\theta_o}$

Since $(2r_i\hat{\theta}_i)/\theta_i \sim \chi^2(2r_i)$, $i=\ell,h$, the distribution of W depends only on r_ℓ, r_h and S^*_ℓ and does not depend on the sample size n_ℓ or n_h. Yet, the duration of the life test is governed by the sample size n_i $(\geqslant r_i)$. For fixed r_i, a larger n_i would lead to a shorter expected experiment time.

Clearly there are many pairs of (r_ℓ, r_h) which satisfy (2.6.9) and (2.6.10) Here, we are particularly interested in the pair whose sum $r_\ell + r_h$ is minimized. This criterion is especially important if the life test is destructive one and the items under testing are expensive. In such situations, it is usually desirable to minimize the number of items that will be destroyed.

2.7 NEW FORMULAE

New formulae for computing the probabilities in (2.6.9) and(2.6.10)are obtained using an explicit expression for the c.d.f of a Chi-squared random variable with an even number of degrees of freedom; namely, if $Y \sim \chi^2(2K)$, then the c.d.f of Y evaluated at y can be written explicitly as

$$F(y;2K) = 1-e^{-y/2}\left[1+\sum_{j=1}^{k-1}(y/2)^j/j!\right],\ y\geqslant 0 \quad ----- (2.7.1)$$

Let U_1 and U_2 be two independent random variables $U_1 \sim \chi^2(2r_l)$, and $U_2 \sim \chi^2(2r_h)$. Then $W \sim U_1^p/U_2^q$.

Hence we can write(2.6.9) as

$$\alpha = \Pr\left[\frac{U_1^p}{U_2^q} \leqslant \frac{(2r_l)^p C_o}{(2r_h)^q}\right]$$

$$= \mathrm{pr}\left[U_1^p \leqslant (2r_l)^p\ C_o\ \left(\frac{U_2}{(2r_h)^q}\right)^q\right]$$

$$= \mathrm{pr}\left[U_1 \leqslant (2r_l)\ (C_o)^{1/p}\left(\frac{U_2}{(2r_h)^{q/p}}\right)^{q/p}\right]$$

$$= \int_0^\infty F\ (\overset{*}{U}_o,\ 2r_l)\ g\ (U;2r_h)du \quad ---- (2.7.2)$$

Where

$$\overset{*}{U}_o = (2r_l)\ (C_o)^{1/p}[U/(2r_h)]^{q/p}$$

$F\ (\overset{*}{U}_o;\ 2r_l)$ is defined by(2.7.1)and $g\ (U;2K)$ is the p.d.f of a $\chi^2(2k)$ random variables. That is,

$$g\ (U;2K) = \frac{U^{k-1}e^{-u/2}}{2^k\ \Gamma K}\ ,\ U\geqslant o \quad ---- (2.7.3)$$

Therefore,

$$F\,(U_o^*;\ 2r_l) = 1-\bar{e}^{U_o^*/2}[1+\sum_{j=1}^{r_l-1} \frac{(U_o^*)^j}{(2)^j}/j!\ ]$$

$$= 1-e^{-(r_l)C_o^{1/p}[U/(2r_h)]^{q/p}}[\ 1+\sum_{j=1}^{r_l-1} (r_l)C_o^{1/p}[U/2r_h]^{q/p}/$$

Setting U = 2t

$$F(U_o^*;\quad 2r_l) = 1-e^{-(r_l)C_o^{1/p}[t/r_h]^{q/p}}[1+\sum_{j=1}^{r_l-1} (r_l)C_o^{1/p}$$

$$[t/r_h]^{q/p}/j!]$$

$$F(U_o^*\ ;\ 2r_l) = 1-e^{-y_o^*}\ [1+\sum_{j=1}^{r_l-1} y_o^*/j!]$$

Where $y_0^* = r_l C_o^{1/p}(t/r_h)^{q/p}$

Now, $g(U;\ 2r_h) = \dfrac{U^{r_h-1}\ e^{-U/2}}{2^{r_h}\ \Gamma_{r_h}}$

$$= \frac{(2t)^{r_h-1}e^{-t}}{2^{r_h}\ \Gamma r_h} \qquad = \qquad \frac{t^{r_h-1}e^{-t}}{2\,\Gamma r_h}$$

So,
$$F(U_o^*;\ 2r_l)\ g\ (U;2r_h) = \left[1-e^{-y_o^*}[1+\sum_{j=1}^{r_l-1} y_o^*/j!]\right]\frac{t^{r_h-1}e^{-t}}{2\ \Gamma r_h}$$

We can write (2.6.9) explicitly as

$$\alpha = 1 - \int_0^\infty \frac{t^{r_h - 1}}{\Gamma r_h} e^{-(t+y_0^*)} \left[1 + \sum_{j=1}^{r_l - 1} \frac{(y_0^*)^j}{1.2,..j}\right] dt \quad \text{----} \ (2.7.4)$$

$\because$ U=2t,du=2dt

Similarly, the power function on the r.h.s of (2.6.10) can be written as

$$r\ (\theta_1) = \int_0^\infty F\ (U_1^*,\ ;\ 2\ r_l)\ g\ (U;\ 2r_h)du \quad \text{------} \ (2.7.5)$$

Where

$$U_1^* = (2r_l)\ (C_0/\emptyset)^{1/p}[U/(2r_l)]^{q/p}$$

Setting U=2t du = 2dt in (2.7.5), we can similarly, Write (2.6.10) explicitly as

$$1-\beta \leq 1 - \int_0^\infty \frac{t^{r_h - 1}}{\Gamma r_h} e^{-(t+y_1^*)} \left[1 + \sum_{j=1}^{r_l - 1} \frac{(y_1^*)^j}{1,2....j}\right] dt \quad \text{----} \ (2.7.6)$$

Where $y_1^* = r_l\ (C_0/\emptyset)^{1/p}(t/r_h)^{q/p}$

We need to truncate the integral at some finite value while computing the improper integral in (2.7.4) or (2.7.6) However, following error bound formula helps us to assess the numerical error caused by this truncation.

For example in evaluating (2.7.4), if the integration range is from 0 to M, then the error is

$$\epsilon_M = \int_M^{\infty} F\,(U_0^*\,;\,2r_l)\; g\,(U\,;\,2\,r_h)du$$

$$\leq \int_M^{\infty} g(U\,;\,2r_h)\;du = e^{-M}[\,1+\sum_{j=1}^{r_h-1}\frac{M^j}{1,2..j}\,] \quad ---(2.7.7)$$

For any specified $\epsilon>0$, one can choose a large M So that $\epsilon_M<\epsilon$. Thus, we can evaluate the improper integrals as accurately as desired.

2.8 ALGORITHM OUTLINE

Step 1 Fix the values of S_l^*, α, β and $\emptyset$

Step 2 Choose a lower bound $r_l^{(1)}$, for the optimal r_l. Note that the optimal r_l increases as S_l^* increases, or as α or β decreases, with other parameters kept fixed.

Step 3 Set $r_l = r_l^{(1)}$ and $r_h = r_l^{(1)}$. Search for C_0 to satisfy (2.7

Step 4 Check whether or not the triplet (r_l, r_h, C_0) satisfies (2.7.5): if yes, go to (b); if not, go to (a).

(a) increase r_h by 5, find C_0, then check if (2.7.5) is satisfied: if yes, go to (a^*); if not, increase r_h by another 5 each time until (2.7.5) is satisfied, then go to (a^*).

(a^*) decrease r_h by 1 until the smallest r_h satisfying (2.7.5) is found. Go to Step 5.

(b) decrease r_h by min (r_h - 1,5), find C_o, then check if (2.7.5) is satisfied: if yes, decrease r_h further by min (r_h-1,5) until (2.7.5) is not satisfied, then go to (b^*); if not, go to (b^*).

(b^*) increase r_h by 1 until the smallest r_h satisfying (2.7.5) is found.

Step 5 Suppose we have computed j pairs of (r_l, r_h) satisfying (2.7.4) and (2.7.5)

Let S(i) denote the sum of the ith pair, $S(i)=r_l^{(i)} + r_h^{(i)}$. If $j \leq 2$, replace r_l by r_l+5, then go to Step 4; if $j \geq 3$, go to next step.

Step 6 For each j (≥ 3), check the pattern of the sums, S(j-1) and S(j):

(a) If $S(j-1) \geq S(j)$, increase $r_l^{(j)}$ by 5, then go to Step 4:

(b) if $S(j-1) < S(j)$, then

(b1) If $j=3$ and if $S(1) \leq S(2)$, conclude $r_l = r_l^{(1)}$ and $r_h = r_h^{(1)}$;

If $j=3$ but $S(1) > S(2)$, conclude that the optimal r_l is in the interval $[r_l^{(1)}, r_l^{(3)}]$, and search for the optimal r_l in this interval and its corresponding r_h and C_0. Go to end.

(b2) if $j>3$, search backward by decreasing $r_l^{(j-2)}$ by 1 each time, for k times, until the sum $S(j-2-k) > S(j-2)$. Then search for the optimal r_l in the interval $[r_l^{(j-2-k)}, r_l^{(j)}]$. Go to end.

Step 7 End

CHAPTER - III

A SAMPLING PLAN FOR SELECTING THE MOST RELIABLE PRODUCT UNDER THE ARRHENIUS ACCELERATED LIFE TEST MODEL

3.1 INTRODUCTION

To select the most reliable product from among several competing products is one of the main problems that the decision maker usually faces at the research and development stage. However, for highly reliable designs, it is very difficult to measure product reliability. Since it may take a long time to perform life testing under normal operating condition.

In order to estimate the product reliability in a short time, we use the ALT method. The products are tested at higher stresses and the resultsare extrapolated, by an assumed model, to estimate the reliability under normal operating condition. When "temperature" is the accelerated factor under consideration, the Arrhenius reaction rate model is often used to describe the relationship of the product parameter (such as failure rate) as a function of operating stress (temperature). Thus, the life of some products and materials in a temperature - accelerated test is described with a weibull distribution. For example, Nelson has used it for capacitor dielectric and for insulating tape. (Nelson 1990, p.82).

Chang, Huang and Tseng (1992) proposed an intuitive rule for selecting the most reliable design under Type - II ALT. One advantage of this rule is that it has a very clear and simple expression; but it requires heavy numerical computation to obtain a sampling plan. Besides, the information contained in the observed data is not efficiently used. To overcome these drawbacks, we propose a modified likelihood ratio (MLR) selection rule which is obtained by the MLR Principle.

3.2 PROBLEM FORMULATION

Let K competing products be denoted by $\pi_1, \ldots, \pi_k$. For $K \geq i \geq 1$, the reliability function of π_i under stress S_o is denoted by $R_i\ (t, S_0)$. S_0 denotes the normal operating condition. At time t^* the product π_i is said to be most reliable if

$$R_i\ (t^*, S_0) = \max_{1 \leq l \leq k} R_l\ (t^*, S_0) \qquad \text{---------} \quad (3.2.1)$$

where t^* is a specific constant (e.g., one year warranty period) which was predetermined by the experimenter, who is only interested in selecting the most reliable product at t^*.

Suppose the life testing was conducted at m values of accelerated stresses $[S_j]_{j=1}^{m}$ where $(S_0 \leqslant)\ S_1 \leqslant \ldots\ldots \leqslant S_m$. It is assumed that life - stress relation follows a weibull-Arrhenius model. That is, under stress S_j the life of product π_i follows a Weibull distribution with a known shape parameter β_i and an unknown product characteristic life (Scale parameter) θ_{ij}. Thus, under stress S_j the reliability function of π_i can be expressed as

$$R_i\ (t,\ S_j) = \exp\ [-(t/\theta_{ij})^{\beta_i}]\ \text{for } t > 0$$

The relationship of stress S_j and θ_{ij} can be expressed as

$$\theta_{ij} = \exp\ (A_i - B_i / S_j) \qquad \text{-------} \quad (3.2.2)$$

Where (A, B_i) denote the unknown parameters of product π_i in the Arrhenius model. To perform an ALT, for each combination of $(\pi_i,\ S_j)$ there are n_{ij} units which are put on test. As soon as r_{ij} failures occur the experiment of $(\pi_i,\ S_j)$ terminates and we record the ordered failure data $Y_{ij(1)} \leq \cdots \leq Y_{ij}\ (r_{ij})$

The standardized stress V_j (Nelson and Meener, 1978) is defined as

$$V_j = \frac{(1/S_j) - (1/S_m)}{(1/S_0) - (1/S_m)} \qquad \text{----------} \quad (3.2.3)$$

It is easily seen that $1 = V_0 > V_1 > \ldots\ldots\ldots > V_m = 0$.

Equation (3.2.2) can be rewritten as

$$\ln\theta_{ij} = \alpha_{io} + \alpha_{i1} V_j \quad \text{----------- (3.2.4)}$$

Where $\alpha_{i1} = - B_i (1/S_0 - 1/S_m)$

and $\alpha_{io} = A_i + B_i\ (-1/S_m)$

Let $Z_{ij(l)} = \beta_i\ [\ln\ (Y_{ij(l)}) - \alpha_{io} - \alpha_{i1}\ V_j]$. Then, we can express the likelihood function of the ith product as

$$\prod_{j=1}^{m} \left[\prod_{l=1}^{r_{ij}} \beta_i\ f\ (Z_{ij(l)}) \right] \left[[1 - F(Z_{ij}(r_{ij}))]^{n_{ij} - r_{ij}} \right] \text{-- (3.2.5)}$$

Where F(.) & f(.) denote the c.d.f and p.d.f for the standard extreme distribution respectively. The MLE of α_{io} and α_{i1}, $(\hat{\alpha}_{io}, \hat{\alpha}_{i1})$ can be obtained by solving

$$\sum_{j=1}^{m} r_{ij} - \sum_{j=1}^{m} \left[\sum_{l=1}^{r_{ij}} \exp\ (Z_{ij(l)}) \right.$$

$$\left. + (n_{ij} - r_{ij})\ \exp\ (Z_{ij}(r_{ij}))\ \right] = 0 \quad \text{-------- (3.2.6)}$$

and

$$\sum_{j=1}^{m} r_{ij}\ V_j - \sum_{j=1}^{m} V_j \left[\sum_{l=1}^{r_{ij}} \exp\ (Z_{ij(l)}) \right.$$

$$\left. + (n_{ij} - r_{ij})\ \exp\ (Z_{ij}(r_{ij})) \right] = 0 \quad \text{--------- (3.2.7)}$$

Consequently (the MLE of θ_{io}) $\hat{\theta}_{io}$ can be obtained from the following equation

$$\hat{\theta}_{io} = \exp\left(\hat{\alpha}_{io} + \hat{\alpha}_{i1}\right) \qquad \text{----------} \ (3.2.8)$$

By the assumption and results mentioned, we have the following lemma

LEMMA 1. $\ln \hat{\theta}_{io}^{\beta_i}$ is asymptotically normally distributed with mean $\ln \theta_{io}^{\beta_i}$ and variance σ_{io}^2, where

$$\sigma_{io}^{2} = \frac{\left(\sum_{j=1}^{m} r_{ij} V_j^2\right) - 2\left(\sum_{j=1}^{m} r_{ij} V_j\right) + \left(\sum_{j=1}^{m} r_{ij}\right)}{\left(\sum_{j=1}^{m} r_{ij}\right)\left(\sum_{j=1}^{m} r_{ij} V_j^2\right) - \left(\sum_{j=1}^{m} r_{ij} V_j\right)^2} \qquad \text{----} \ (3.2.9)$$

Proof The fisher information matrix of $(\alpha_{io}, \alpha_{i1})$ can be expressed as

$$\beta_i^2 \begin{bmatrix} \sum_{j=1}^{m} r_{ij} & \sum_{j=1}^{m} r_{ij} V_{ij} \\ \sum_{j=1}^{m} r_{ij} V_{ij} & \sum_{j=1}^{m} r_{ij} V_{ij}^2 \end{bmatrix}$$

Since $\ln \hat{\theta}_{io} = (\hat{\alpha}_{io} + \hat{\alpha}_{i1})$, $\ln \hat{\theta}_{io}^{\beta_i}$ is asymptotically normally distributed with mean $\ln \hat{\theta}_{io}^{\beta_i}$ and variance σ_{io}^2

3.3 MLR SELECTION RULE AND SAMPLING PLAN

Without loss of generality, we assume that $t^* = 1$. According to kingston and Patel (1980), the observed failure time can be scaled So that $t^* = 1$. Then (3.2.1) can be rewritten as

$$(\theta_{io})^{\beta_i} = \max_{1 \le l \le k} (\theta_{lo})^{\beta_l} \quad \text{------- (3.3.1)}$$

Based on the above ALT data, we propose an MLR selection rule $\delta = (\delta_1, \ldots\ldots \delta_k)$ as follows:

$$\delta_i : \text{Select } \pi_i \text{ if and only if } \prod_{j \neq i}^{k} [\hat{\theta}_{io}^{\beta_i} / \hat{\theta}_{jo}^{\beta_j}] \geq d \quad \text{----- (3.3.2)}$$

If d is too small, then this MLR rule may lead to selecting more than one product. We introduce a procedure to determine the values of $[r_{ij}, n_{ij}]$ and d in order to select the most reliable product.

If the selected π_i is the most reliable product then we call the selection rule δ_i a correct selection (CS). However, the selection rule δ_i is called an error selection if the selected π_i is not a reliable one.

Let P_τ (SC/δ_i) denote the probability of CS of rule δ_i under $\tau = (T_{i1}, \ldots T_{ii-1}, T_{ii+1}, \ldots, T_{ik})$, where Tij denotes the measure of separation of products π_i and π_j. We define $T_{ij} = \ln [\ln R_j (t^*)/\ln R_i (t^*)]$. As $t^*=1$, T_{ij} can be expressed as $\ln (\theta_{io}^{\beta_i}/\theta_{jo}^{\beta_j})$. Thus, the ith preference region Ω_i, may be defined as

$$\Omega_i = [\tau/T_{ij} \geqslant \Delta , \text{ for } j \neq i], \Delta>0 \quad \text{-------} \ (3.3.3)$$

Similarly, the indifference region, Ω_0, can be defined as

$$\Omega_0 = [\tau/T_{ij} = 0, \text{ for } j \neq i] \quad \text{---------} \ (3.3.4)$$

It is usually required that the error probability have a maximum value α^* for $\tau \in \Omega_0$ (α^* - condition) and the probability of CS have a minimum value P^* for $\tau \in \Omega_i$ (P^* - condition). That is, the α^* and P^* conditions can be expressed as (Tseng and change, 1989).

$$\sup_{\tau \in \Omega_0} P_\tau(CS/\delta_i) \leq \alpha^* \ \& \ \inf_{\tau \in \Omega_i} P_\tau(CS/\delta_i) \geq P^*$$

In case of $r_{ij} = r_j$ and $n_{ij} = n_j$ for $1 \leq i \leq k$ and $1 \leq j \leq m$, these two conditions are asymptotically approximated, using theorem 3 stated in 3.4 by

$$\phi \left(\frac{\ln d}{\sqrt{K (k-1) \sigma_0^2}} \right) \geqslant (1-\alpha^*) \quad \text{-----} \ (3.3.5)$$

and $$\phi\left(\frac{\ln d - (k-1)\Delta}{\sqrt{K(k-1)\,\sigma_o^2}}\right) \leq (1 - P^*) \quad \text{-------} \ (3.3.6)$$

Where ϕ (.) denotes the cdf of standard Normal distribution. Moreover, a decision maker usually wants to control the time - saving factor of life testing at a Specific level ρ^* (ρ^* - condition). So, the time saving factor can be defined as follows (Tseng and Chang, 1989):

$$E[Y_{ij}(r_{ij})] \leq \rho^* E[Y_{ij}(n_{ij})] \quad \text{------} \ (3.3.7)$$

for each combination of (π_i, S_j)

Let R_a and R_b denote the reliability of (K-1) less reliable products and most reliable product at normal operating condition, respectively. Then Δ can be expressed as $\ln(\ln R_a / \ln R_b)$. Set $r_j = ra_j$, where $[a_j]_{j=1}^{m}$ are predetermined. We state an algorithm to compute $[(n_j, r_j)]_{j=1}^{m}$ and d under various combinations of K, P^*, α^*, ρ^*, Δ, ($a_1, \ldots, a_m$) and stresses $S_1 < \ldots\ldots < S_m$ ($1 = V_o \geq V_1 \geq \ldots\ldots \geq V_m = 0$)

Step 1. Start with r = 1

Step 2. Multiplying (a_1, ... a_m) by r we get the associated value of (r_1, ... r_m).

Step 3. Compute d by (3.3.6)

Step 4. Check by (3.3.5). If it holds, then $(r_1, \ldots, r_m)$ is a feasible solution, go to step 5. Otherwise, set $r = (r+1)$ and return to step 2.

Step 5. Compute the corresponding sample size $(n_1, \ldots, n_m)$ from (3.3.7).

THEOREM 1 The α^* condition and P^* condition are equivalent to (3.3.5) and (3.3.6) respectively.

Proof

$$P_\tau(CS/\delta_i) = P_\tau\left[\prod_{j\neq i}\left((\hat{\theta}_{io})^{\beta_i}/(\hat{\theta}_{jo})^{\beta_j} \geq d\right)\right]$$

$$= P_\tau\left[(K-1)\ln(\hat{\theta}_{io})^{\beta_i} - \sum_{j\neq i}^{k}\ln(\hat{\theta}_{jo})^{\beta_j} \geq \ln d\right]$$

$$= P_\tau\left(Z \geq \frac{\ln d - \sum_{j\neq i}[\ln(\theta_{io})^{\beta_i} - \ln(\theta_{jo})^{\beta_j}]}{\sqrt{K(k-1)\sigma_0^2}}\right)$$

Where Z denotes the standard normal density. Thus α^* - condition and P^* - condition can be expressed by (3.3.5) and (3.3.6) respectively.

3.4. THEORETICAL DERIVATION OF THE MIR SELECTION RULE

Let K independent products and m different levels of accelerated stresses be denoted by $\pi_1, \ldots \pi_k$ and $S_1 \leq S_2 \leq \ldots \leq S_m$ respectively. For each cell of (π_i, S_j), n_{ij}. Items are exposed to a life test. Let the first r_{ij} ordered failure data be denoted by $Y_{ij\ (1)} \leq \cdots \leq Y_{ij\ (r_{ij})}$. Let the life distribution for each cell of (π_i, S_j) follows a weibull distribution with shape parameter β_i and scale parameter θ_{ij}. The relationship between θ_{ij} and S_j in the Arrhenius model can be expressed as (3.2.2) and the MLE of θ_{io} can be computed by (3.2.8).

Since $T_{ij} = \ln(\theta_{io}^{\beta_i}/\theta_{jo}^{\beta_j})$, we define $t_{ij} = \ln(\hat{\theta}_{io}^{\beta_i}/\hat{\theta}_{jo}^{\beta_j})$ under the assumption that $r_{ij} = r_j$, for $j = 1,\ldots m$ and with the notation stated previously, we have the following result:

THEOREM 2 The joint p.d.f of $T = (t_{i1}, t_{ii-1}, t_{ii+1}, \ldots, t_{ik})$ under τ is

$$h_\tau(t) = \frac{1}{(2\pi)^{k-1/2}|\Sigma_T|^{1/2}} \exp\left[-\tfrac{1}{2}(t-\tau)' \Sigma_T^{-1}(t-\tau)\right] \tag{3.4.1}$$

Where

$$\Sigma_T = \sigma_0^2 \begin{vmatrix} 2 & 1 & \cdots & \cdots & 1 \\ 1 & 2 & 1 & & \vdots \\ \vdots & 1 & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 & 2 \end{vmatrix}_{(K-1)\times(K-1)}$$

and

$$\sigma_0^2 = \frac{\left(\sum_{j=1}^m r_j V_j^2\right) - 2\left(\sum_{j=1}^m r_j V_j\right) + \left(\sum_{j=1}^m r_j\right)}{\left(\sum_{j=1}^m r_j\right)\left(\sum_{j=1}^m r_j V_j^2\right) - \left(\sum_{j=1}^m r_j V_j\right)^2}$$

Proof **Let**

$$Y = \begin{bmatrix} \beta_1 \ln \hat{\theta}_{10} \\ \vdots \\ \beta_k \ln \hat{\theta}_{k0} \end{bmatrix}, \quad \mu = \begin{bmatrix} \beta_1 \ln \theta_{10} \\ \vdots \\ \beta_k \ln \theta_{ko} \end{bmatrix} \text{ and}$$

$$\Sigma = \begin{vmatrix} \sigma_{10}^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_{ko}^2 \end{vmatrix}$$

By Lemma 1, it is seen that Y is asymptotically distributed with N (μ, Σ)

Since, T = HY, where

$$\begin{array}{c} \downarrow \text{ ith column} \\ \left[\begin{array}{cccccccc} -1 & & & 1 & & & & \\ & & & \vdots & & 0 & & \\ & -1 & & \vdots & & & & \\ & & \ddots & & & & & \\ & & -1 & 1 & & & & \\ \hline & & & 1 & -1 & & & \\ & & & \vdots & & -1 & & \\ & 0 & & \vdots & & & \ddots & \\ & & & 1 & & & & -1 \end{array}\right] \end{array}$$

So, T is asymptotically distributed with N (μ_T, Σ_T), Where

$$\mu_T = H\mu = \tau = \begin{bmatrix} \beta_i \ln\theta_{io} - \beta_1 \ln\theta_{10} \\ \vdots \\ \beta_i \ln\theta_{io} - \beta_k \ln\theta_{ko} \end{bmatrix}$$

and

$$\Sigma_T = H\Sigma H' = \left|\begin{array}{ccc:ccc} \sigma_{10}^2 + \sigma_{i0}^2 & \sigma_{i0}^2 & & & \sigma_{i0}^2 & \\ & & & & & \\ \sigma_{i0}^2 & \sigma_{i-10}^2 + \sigma_{i0}^2 & & & & \\ \hdashline & & & \sigma_{i+10}^2 + \sigma_{i0}^2 & & \sigma_{i0}^2 \\ & & & & \ddots & \\ & \sigma_{io}^2 & & \sigma_{i0}^2 & \sigma_{ko}^2 + \sigma_{io}^2 & \end{array}\right|$$

underthe condition that $r_{ij} = r_j$, we have $\sigma_{io}^2 = \sigma_o^2$, $1 \le i \le k$

So Σ_T can be reduced to

$$\Sigma_T = \sigma_0^2 \begin{vmatrix} 2 & 1 & \cdots & \cdots & 1 \\ 1 & 2 & 1 & & \vdots \\ \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 2 \end{vmatrix}_{(k-1)\times(k-1)}$$

In order to construct a suitable selection rule, we consider a family of hypotheses as follows:

$$H_0 : \tau \in \Omega_o \text{ Vs } H_i : \tau \in \Omega_i,\ i = 1, \ldots, K.$$

The MLR Selection rule $\delta = (\delta_1, \ldots, \delta_k)$ can be defined as

$$\delta_i : \text{Select } \pi_i \text{ if and only if } \frac{\inf_{\tau \in \Omega_i} h_\tau(t)}{\sup_{\tau \in \Omega_o} h_\tau(t)} \geq C \text{ --- } (3.4.2)$$

Where 'C' is a constant. The following result follows from Theorem 2

THEOREM 3 The MLR selection rule can be expressed approximately as

$$\delta_i : \text{select } \pi_i \text{ if and only if } \prod_{j \neq i}^{K} [\hat{\theta}_{io}^{\beta_i} / \hat{\theta}_{jo}^{\beta_j}] \geq d \quad (3.4.3)$$

where d is a constant to be determined

Proof

Since $\inf_{\tau \in \Omega_i} h_\tau(t)$ depends on the value of t, it is rather complicated to derive the MLR rule. Instead of using Ω_i in (3.3.3), we restrict our attention to Ω_i^*

Where

$$\Omega_i^* = \{\tau / T_{ij} = \Delta , \text{ for } j \neq i\}, \ \Delta > 0.$$

Then (3.4.2) can be approximately expressed as

$$\{\exp\left(-\tfrac{1}{2}\,(t-\Delta)' \,\Sigma_T^{-1}\,(t-\Delta)\right)/\exp\left[-\tfrac{1}{2}(t-0)' \cdot \Sigma_T^{-1}\,(t-0)\right]\} \geq C.$$

It can rewritten as

$$2t' \,(\Sigma_T)^{-1}\Delta - \Delta' \,(\Sigma_T)^{-1}\Delta \geq C_1,$$

Where C_1 is a constant. Since

$$t' \,(\Sigma_T)^{-1}\Delta = \frac{\Delta}{\sigma_0^2}\,\frac{1}{k}\,[\sum_{j \neq i}^{k} t_{ij}] \text{ and } \Delta' \,(\Sigma_T)^{-1}\Delta = \frac{\Delta^2}{\sigma_0^2}\left(\frac{k-1}{k}\right)$$

(3.4.2) can be expressed as

$$\sum_{j \neq i} t_{ij} \geq C_2,$$

Where C_2 is a constant. Since $t_{ij} = \ln\,(\hat{\theta}_{i0}^{\beta_i} / \hat{\theta}_{jo}^{\beta_j})$, we have

$$\prod_{j \neq i}^{K} \,[\hat{\theta}_{io}^{\beta_i} / \hat{\theta}_{j0}^{B_j}] \geq d,$$

Where d is a constant to be determined.

CHAPTER - IV

PLANNING ACCELERATED LIFE TESTS FOR SELECTING THE MOST RELIABLE PRODUCT

4.1 INTRODUCTION

At the research and development stage, a decision maker usually faces the problem of selecting the most reliable product design from several competing designs.

ALT is a commonly used method for comparing and estimating the lifetime of highly reliable products in a short time. Products are tested at higher stress such as temperature, voltage, vibration etc. and results are extrapolated using an assumed statistical model to estimate the product life at normal operating stress. More explicitly, the life of certain products is described with a Weibull life distribution whose characteristic life is a log-linear function of stress (Nelson, 1990). The insulating tape, capacitor dielectric, and quartz oscillator with temperature as the accelerating variable, are some of the examples for weibull-Arrhenius model. Applications for Weibull- inverse power models are electrical insulation, ball bearings and metal fatigue with voltage, load and mechanical stress as the accelerating variable respectively.

To select the most reliable design from several highly reliable designs, Chang et al. (1992) and Tseng et al (1994) have proposed some selection rules under type-II ALT plan.

Here under the case of known Weibull shape parameters the optimum test plans for both type-I and type-II censoring are derived by minimizing the asymptotic variance of estimated quantiles at design stress. Based on life data from these plans, we propose a selection rule to achieve a stated goal. The selection rule needs sample size and censoring time (or number of failures) and both of these are computed under a predetermined time saving factor and a minimum probability of correct selection. By using a cost criterion, we compare the relative efficiency of these two censoring plans.

4.2 THE LIFE - STRESS MODEL

4.2.1 Weibull - Log - Linear Model:

The assumptions of this model are as follows:

a) At any stress the product life has a Weibull distribution. The Weibull reliability function is

$$R(t) = e^{-(t/\theta)^{\beta}}, \quad t \geq 0$$

Where $\beta>0$ and $\theta>0$ are the Weibull shape & scale parameters respectively.

b) The Weibull Characteristic life is a log-linear function of X (s).

$$\ln\theta = r_0 + r_1 x\ (S)$$

and

$$X\ (S) = \begin{cases} \ln s & \text{if inverse power model is assumed} \\ 1/s & \text{if Arrhenius model is assumed} \end{cases} \qquad \text{---- } (4.2.1.1)$$

Where r_1 and r_0 are unknown parameters to be estimated.

c) The Weibull shape parameter β is independent of stress (a constant for any stress).

4.2.2 Censoring Mechanism:

To shorten the time of life testing, we make use of censoring plans.

a) **Type-I Censoring:**

Each unit is run for a predetermined time. The censoring time is fixed and the number of failures in that fixed time is random.

b) **Type-II Censoring:**

The units are tested simultaneously until a predetermined number of them fail. The test is stopped when a specified number of failures occur. The time to that fixed number of failures is random.

4.2.3 The Optimization Criterion:

Various criteria for determining optimal ALT plans have been described by Nelson and Kielpinski (1976).

4.3 PROBLEM FORMULATION

Suppose K available product designs be denoted by $\pi_1, \ldots\ldots\ldots, \pi_k$.

Let the normal use condition (stress) of those designs be denoted by S_o.

For $1 \leq l \leq k$, $R_l\ (t, S_o)$ denotes the reliability function of π_l under stress S_o.

At time t^*, the design π_i is said to be the most reliable design; if.

$$R_i\ (t^*, S_o) = \max_{1 \leq l \leq k} R_l\ (t^*, S_o) \qquad \text{-------} \qquad (4.3.1)$$

From these K available designs the decision maker's aim is to select the most reliable design.

For highly reliable products, there may be only a few or even no failures observed under S_o. To overcome this problem, ALT is used.

Suppose the tests are conducted at m values of higher stresses.

$$\{S_j\}_{j=1}^{m} \quad \text{and} \quad S_0 < S_1 < \ldots\ldots\ldots < S_m.$$

It is assumed that the lifetime of design π_i under stress S_j follows a Weibull distribution with an unknown characteristic life (Scale parameter) θ_{ij} and a shape parameter β_i, where θ_{ij} with S_j follows a log-linear model.

This can be expressed as

$$\ln \theta_{ij} = r_{io} + r_{i1} X (S_j) \quad \text{------} \quad (4.3.2)$$

Where, As in equation (4.2.1.1) X (S) is a function of stress S and r_{i1} & r_{io} are unknown parameters of design π_i .

To perform an ALT there are n_{ij} units which are put on test for each combination of (π_i, S_j). Using censoring plan (type I or type II) for each combination-as soon as the censoring time n_{ij} (or the number of failures r_{ij}) is reached, the experiment terminates. To select the most reliable design based on these life testing data, the typical decision problems are:

(a) Which is the better censoring plan?

(b) For performing ALT, how many stresses should be used?

(c) For each combination of (π_i, S_j), how many observations n_{ij} should be taken?

(d) For each combination of (π_i, S_j), what is the optimal censoring time n_{ij} or the optimal number of failures r_{ij}?

(e) To achieve the goal of the experimenter, how a suitable selection rule is to be constructed?

4.4 OPTIMAL ACCELERATED LIFE TEST PLAN

To estimate the unknown parameters under both type-I and type-II censoring we use maximum likelihood (ML) method.

The standardized stress V_j is defined as follows:

$$V_j = (\ln S_m - \ln S_j)/(\ln S_m - \ln S_o), \quad 0 \le j \le m \quad \text{--- (4.4.1)}$$

It is easily seen that $V_m = 0$ and $V_o = 1$, while $1 > V_j > 0$ for $(m-1) \ge j \ge 1$.

The relation in equation (4.3.1) can be rewritten as

$$\ln \theta_{ij} = \alpha_{io} + \alpha_{i1} V_j \quad \text{---- (4.4.2)}$$

Where

$\alpha_{i1} = r_{i1} (\ln S_m - \ln S_o)$ and $\alpha_{io} = (r_{io} - r_{i1} \ln S_m)$

Let $[T_{ij\ell}]_{\ell=1}^{n_{ij}}$ denote a set of observations for the combination of (π_i, S_j) and $Z_{ij\ell} = \beta_i (\ln T_{ij\ell} - \alpha_{io} - \alpha_{i1} V_j)$.

A standard extreme distribution is followed by $Z_{ij\ell}$, for i, j, ℓ where $K \ge i \ge 1$, $m \ge j \ge 1$ and $n_{ij} \ge \ell \ge 1$

Consider a sample that may be type-I or type-II censored involving observations on the lifetimes of n_{ij} individuals for each combination of (π_i, S_j). We denote both censoring time and standardized lifetime as $Z_{ij\ell}$ $(\ell = 1, .., n_{ij})$. Let C_{ij} be the set for which $Z_{ij\ell}$ is a standardized censoring time and D_{ij} be the set of individuals for which $Z_{ij\ell}$ is an observed lifetime. For the i-th design the likelihood function can be expressed as:

$$\prod_{j=1}^{m} [\prod_{\ell \in D_{ij}} \beta_i \, \phi \, (Z_{ij\ell}) \prod_{\ell \in C_{ij}} Q \, (Z_{ij\ell})] \quad \text{-----} \quad (4.4.3)$$

Where Q (z) and ϕ (z) denote reliability function and p.d.f for the standard extreme distribution, respectively.

The MLE for α_{io} and α_{i1}, ($\hat{\alpha}_{io}$, $\hat{\alpha}_{i1}$) can be solved by

$$\sum_{j=1}^{m} r_{ij} = \sum_{j=1}^{m} [\sum_{\ell \in D_{ij}} e^{z_{ij\ell}} + \sum_{\ell \in C_{ij}} e^{Z_{ij\ell}}] \quad \text{--} \quad (4.4.4)$$

$$\sum_{j=1}^{m} r_{ij} V_j = \sum_{j=1}^{m} V_j [\sum_{\ell \in D_{ij}} e^{Z_{ij\ell}} + \sum_{\ell \in C_{ij}} e^{Z_{ij\ell}}] \quad \text{--} \quad (4.4.5)$$

Where r_{ij} is the number of individuals in D_{ij}.

Since from equation (4.4.1), we have $\ln \hat{\theta}_{io} = (\hat{\alpha}_{io} + \hat{\alpha}_{i1})$ for all $K \geq i \geq 1$, so we can obtain the following lemma.

LEMMA 1.

$\ln \hat{\theta}_{io}$ is asymptotically normally distributed with mean $\ln \theta_{io}$ and variance.

$$\frac{1}{\beta_i^2}\left[\frac{(\sum_j w_{ij} v_j^2) - 2(\sum_j w_{ij} v_j) + \sum_j w_{ij}}{(\sum_j w_{ij})(\sum_j w_{ij} v_j^2) - (\sum_j w_{ij} v_j)^2}\right] \text{------- } (4.4.6)$$

Where

$$W_{ij} = \begin{cases} n_{ij} M_{ij} & \text{for type - I censoring} \\ r_{ij} & \text{for type - II censoring} \end{cases} \text{------- } (4.4.7)$$

and

$$M_{ij} = 1 - e^{-(n_{ij}/\theta_{ij})^{\beta_i}}$$

LEMMA 2. For both type-I and type-II censoring, the necessary condition for minimizing

$\mathrm{Var}(\ln \hat{\theta}_{io})$ is $V_1 = V_2 = \ldots\ldots = V_{m-1}$

From this lemma, it follows that $m \geq 3$ are non-optimal. This means that only two higher stresses (m-=2) are needed to perform accelerated life test.

For simplicity, let L and H denote the low and high stresses. Now, let P_{iL} (P_{iH}) denote the proportion of the sample size allocated to the low (high) stress, and let q_{iL} (q_{iH}) denote the proportion of the number of failures allocated to the low (high) stress. Suppose that n_{i0} and r_{i0} denote the total sample size and the number of failures which are needed by the i-th design (population). Then $n_{ij}=n_{io}P_{ij}$ & $r_{ij}=r_{io}q_{ij}$ for j=L, H and Equation (4.4.5) can be rewritten as

$$\mathrm{Var}(\ln\hat{\theta}_{io}) = \begin{cases} \dfrac{1}{\beta_i^2 V_L^2\, n_{i0}}\left\{\dfrac{1}{P_{iL}M_{iL}} + \dfrac{(1-V_L)^2}{(1-P_{iL})M_{iH}}\right\} & \text{for type-I censoring} \\[2ex] \dfrac{1}{\beta_i^2\, V_L^2\, r_{i0}}\left\{\dfrac{1}{q_{iL}} + \dfrac{(1-V_L)^2}{1-q_{iL}}\right\} & \text{for type II censoring} \end{cases} \qquad (4.4.8)$$

It is impossible to find a non-trivial solution (V_L, P_{iL}) $(\mathrm{Or}\,(V_L, q_{iL})) \neq (1,1)$, such that Var $(\ln\hat{\theta}_{i0})$ attains a minimum. Consequently, we shall fix V_L and minimize with respect to P_{iL} (Or q_{iL}).

LEMMA 3. For type-I censoring, the optimal proportion of the sample size allocated to the low stress is

$$P^*_{iL} = \frac{1}{1 + (1-V_L)\sqrt{(M_{iL}/M_{iH})}}$$

For type-II censoring, the optimal proportion of the number of failures allocated to the low stress is

$$q^*_{iL} = \frac{1}{2 - V_L}$$

From Lemma 3, if $M_{iL} = M_{iH}$, then $P^*_{iL} = q^*_{iL}$. Besides, if $r_{i0} = n_{i0}\, M_{iL}$, then $\ln(\hat{\theta}_{i0}^{\beta_i})$ is asymptotically normally distributed with mean ln $(\hat{\theta}_{i0}^{\beta_i})$ and variance $(1/r_{i0})\,[(2-V_L)/V_L]^2$.

4.5 A SELECTION RULE

Based on the life data as described above, we propose a selection rule as follows:

δ: select design π_i if

$$\hat{R}_i(t^*, S_0) = \max_{1\leq \ell \leq k} \hat{R}_\ell(t^*, S_0), \quad \text{---------} \qquad (4.5.1)$$

Where

$$\hat{R}_\ell(t^*, S_0) = e^{-(t^*/\hat{\theta}_{i0})^{\beta_i}}, \text{ for } 1\leq \ell \leq k.$$

This selection rule is completely specified when the sample size and censoring time (or number of failures) are known. In the following, we develop a procedure to determine these values.

For simplicity, we define the i-th preference region as follows:

$$\Omega_i = \{(R_1, \ldots, R_k) \mid R_i^{\Delta} \geq \max_{\ell \neq i} R_\ell\},\ \Delta > 1. \text{---} \qquad (4.5.2)$$

The selection rule in Equation (4.5.1) gives a correct selection if π_i is the most reliable design and we can correctly select it. Let P_R (CS/δ) denote the probability of a correct (CS) by using the selection rule δ. It is usually required that the probability of CS exceeds a minimum value P^* (referred to as the P^* -condition), that is,

$$\inf_{R\in\Omega_i} P_R\ (CS|\delta) \geq P^*, \qquad \text{------ (4.5.3)}$$

Where P^* is a value predetermined by the decision maker.

To control the accelerated life-testing time within a specified level, we can compute the sample size in terms of the number of failures (refer to Tseng and Wu, 1990) by

$$E(Y_{ijr_{ij}})/E(Y_{ijn_{ij}}) \leq \zeta_{ij}, \qquad \text{------- (4.5.4)}$$

Where $(Y_{ij1},\ldots.Y_{ijn_{ij}})$ denotes the order statistic of $(T_{ij1}\ldots.,\ T_{ijn_{ij}})$, and ζ_{ij} is a fixed constant.

We now state a lemma to compute $E(Y_{ijr_{ij}})$ as follows:

LEMMA 4. For

$$g(a,b,c) = \int_0^1 \frac{[-\ln\ (1-y)]^{1/c} y^{a-1} (1-y)^{b-1}}{\beta(a,b)}\, dy,$$

$$E(Y_{ijr_{ij}}) = g\ (r_{ij}, n_{ij} - r_{ij} + 1,\ \beta_i)$$

We now state two theorems to compute the optimal sampling plan for selecting the most reliable population under both type-I & type-II censoring.

THEOREM 1. For type-II censoring, the sample sizes and number of failures $\{(n_{ij}, r_{ij})\}_{j=L}^{H}$, $1 \le i \le k$, can be solved by using the asymptotic approximation

$$\int_0^1 \prod_{j \neq 1}^{k} [\phi \left(\frac{\sqrt{r_{j0}}}{\sqrt{r_{i0}}} \phi^{-1}(x) + \sqrt{r_{j0}} (\ln \Delta) \left(\frac{V_L}{2-V_L} \right) \right)] \, dx \ge P^* \quad \text{--(4.5.5)}$$

$$r_{ij} = r_{i0} q^*_{ij} , \quad \text{--------- (4.5.6)}$$

and

$$\frac{g\,(r_{ij},\ n_{ij} - r_{ij} + 1,\ \beta_i)}{g\,(n_{ij},\ 1,\ \beta_i)} \le \zeta_{ij} \quad \text{------ (4.5.7)}$$

Where ϕ is the c.d.f of the standard normal distribution.

THEOREM 2. For type-I censoring, the sample sizes and censoring times $\{(n_{ij}, \eta_{ij})\}_{j=L}^{H}$, $1 \le i \le k$, can be solved by using the asymptotic approximation

$$\int_0^1 \prod_{j \neq i}^{k} \left[\phi \left(\frac{\sqrt{r'_{jL}}}{\sqrt{r'_{iL}}} \left(\frac{P^*_{jL}}{P^*_{iL}} \right) \phi^{-1}(X) + \sqrt{r'_{jL}} \,(\ln \Delta)(V_L P^*_{jL}) \right) \right] dX \ge P^*, \quad \text{-------- (4.5.8)}$$

$$\frac{g\,(r'_{iL},\ n_{i0}-r'_{iL}+1, \beta_i)}{g\,(n_{i0}, 1\ ,\ \beta_i)} \leq \zeta_{io} \qquad \text{------ (4.5.9)}$$

$$n_{ij} = n_{i0}\, P^{*}_{ij}, \qquad \text{--------- (4.5.10)}$$

and

$$\eta_{ij} = \xi_j * \theta_{ij}, \qquad \text{---------- (4.5.11)}$$

Where

$$\xi_j = \{ -\ln(1-\frac{r'_{ij}}{n_{i0}}) \}^{1/\beta_i}$$

4.6 COMPARISON BETWEEN TYPE-I AND TYPE-II CENSORING

Mackay (1977) has suggested some criteria for comparing the two censoring plans, for example, the cost of the experiment and the duration of the experiment. Here we have exphasised on the cost of the experiment.

(a) Product's unit cost

If the product's unit cost is very high, then we prefer type-I censoring plan.

(b) Expected life-testing time

To compare the relative efficiency of type-II censoring with type-I censoring, we define a quantity as follows:

$$\ell = \sum_{i=1}^{k} \sum_{j=1}^{m} [E (Y_{ij} \; r_{ij} \;)/n_{ij}] \qquad \text{------ } (4.6.1)$$

ℓ can be reduced to

$$\ell = \sum_{i=1}^{k} \sum_{j=1}^{m} \frac{\int_0^1 (-\ln(1-Y))^{1/\beta_i} \; Y^{r_{ij}-1}(1-Y)^{n_{ij}-r_{ij}} dy}{\beta \; (r_{ij}, n_{ij}-r_{ij}+1)(-\ln(1-r_{ij}/n_{ij}))^{1/\beta_i}} \text{ --}(4.6$$

Where

$$\beta(a,b) = \int_0^1 Y^{a-1}(1-Y)^{b-1} dy$$

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